Second-Order Cosmological Perturbations from Inflation

Viviana Acquaviva[1*], Nicola Bartolo[2,3 †], Sabino Matarrese[2,3] and Antonio Riotto[3]
(1)Dipartimento di Fisica, Università di Pisa, via Buonarroti 2, I-56100, Pisa, Italy
(2) Dipartimento di Fisica di Padova "G. Galilei", Via Marzolo 8, Padova I-35131, Italy
(3) INFN, Sezione di Padova, Via Marzolo 8, Padova I-35131, Italy
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We present the first computation of the cosmological perturbations generated during inflation up to second order in deviations from the homogeneous background solution. Our results, which fully account for the inflaton self-interactions as well as for the second-order fluctuations of the background metric, provide the exact expression for the gauge-invariant curvature perturbation bispectrum produced during inflation in terms of the slow-roll parameters or, alternatively, in terms of the scalar spectral n_S and and the tensor to adiabatic scalar amplitude ratio r. The bispectrum represents a specific non-Gaussian signature of fluctuations generated by quantum oscillations during slow-roll inflation. However, our findings indicate that detecting the non-Gaussianity in the cosmic microwave background anisotropies emerging from the second-order calculation will be a challenge for the forthcoming satellite experiments.

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I. INTRODUCTION

Inflation represents a successful mechanism for the causal generation of primordial cosmological perturbations in the early Universe [1]. These fluctuations are then amplified by the gravitational instability to seed structure formation in the Universe, and to produce Cosmic Microwave Background (CMB) anisotropies. Since the primordial cosmological perturbations are tiny, the generation and evolution of fluctuations during inflation has always been studied within linear theory. On the other hand, there exist physical observables, such as the three-point function of scalar perturbations, or its Fourier transform, the bispectrum, for which a perturbative treatment up to second order is required, in order to obtain a self-consistent result

The importance of the bispectrum comes from the fact that it represents the lowest order statistics able to distinguish non-Gaussian from Gaussian perturbations for which odd-order correlation functions necessarily vanish. An accurate calculation of the primordial bispectrum of cosmological perturbations has become an extremely important issue, as a number of present and future experiments, such as MAP and *Planck*, will allow to constrain or detect non-Gaussianity of CMB anisotropy data with high precision.

So far, the problem of calculating the bispectrum of perturbations produced during inflation has been addressed by either looking at the effect of inflaton self-interactions (which necessarily generate non-linearities in its quantum fluctuations) in a fixed de Sitter background [2], or by using the so-called stochastic approach to inflation [3][1], where back-reaction effects of field fluctuations on the background metric are partially taken into account. An intriguing result of the stochastic approach is that the dominant source of non-Gaussianity actually comes from non-linear gravitational perturbations, rather than by inflaton self-interactions.

In this paper we provide for the first time the computation of the scalar perturbations produced during single-field slow-roll inflation up to second order in deviations from the homogeneous background. We achieve different goals. First, we provide a gauge-invariant definition of the comoving curvature $\mathcal{R}$ at second order of perturbation theory. The importance of the second-order comoving curvature perturbation $\mathcal{R}$ comes from the fact that it allows to compute the three-point correlation function for the primordial scalar perturbations – or its Fourier transform, the bispectrum – which represent the lowest order statistics able to distinguish non-Gaussian from Gaussian perturbations. Secondly, we show that the second-order comoving curvature perturbation is conserved on super-horizon scales, like its first-order counterpart[2]. Third, we obtain the expression for the gauge-invariant gravitational potential bispectrum

[*]Address after Nov. 2002: SISSA/ISAS, Via Beirut 4, I-34014, Trieste, Italy.
[†]Address after Nov. 2002: Astronomy Centre, University of Sussex, Brighton BN1 9QJ, U.K..
[1]See also Ref. [4].
[2]For related results see also Ref. [5].

during inflation, in terms of slow-roll parameters or, equivalently, of the spectral indices of the scalar and tensor power-spectra.

An accurate calculation of the primordial bispectrum of cosmological perturbations has become a crucial issue, as a number of present and future experiments, such as MAP and *Planck*, will allow to constrain or detect non-Gaussianity of CMB anisotropy data with high precision.

The plan of the paper is as follows. In Section II we write the perturbations of the metric for a spatially flat Robertson-Walker background up to second order and we derive consistently the fluctuations of the energy-momentum tensor of a scalar field. In Section III we demonstrate how to find a second-order gauge-invariant definition of the comoving curvature perturbation. Section IV will be devoted to the perturbed Einstein equations up to second order in the metric and in the inflaton fluctuations. We shall explain how to derive the evolution of the curvature perturbation on large scales during a period of cosmological inflation by performing an expansion to lowest order in the slow-roll parameters. Finally, in Section V we draw some concluding remarks relating our findings to the gauge-invariant gravitational potential bispectrum which is the main physical observable which carries information about the primordial non-Gaussianity. We also provide two Appendices where the reader can find various technical details.

II. PERTURBATIONS OF A FLAT ROBERTSON-WALKER UNIVERSE UP TO SECOND ORDER

We first write down the perturbations on a spatially flat Robertson-Walker background following the formalism of Refs. [6,7]. We shall first consider the fluctuations of the metric, and then the fluctuations of the energy-momentum tensor of a scalar field. Hereafter greek indices run from 0 to 3, while latin indices label the spatial coordinates from 1 to 3. If not otherwise specified we will work with conformal time τ, and a prime will stand for a derivative with respect to τ.

A. The metric tensor

The components of a perturbed spatially flat Robertson-Walker metric can be written as

$$\begin{aligned} g_{00} &= -a^2(\tau)\left(1+2\phi^{(1)}+\phi^{(2)}\right)\,,\\ g_{0i} &= a^2(\tau)\left(\hat{\omega}_i^{(1)}+\frac{1}{2}\hat{\omega}_i^{(2)}\right)\,,\\ g_{ij} &= a^2(\tau)\left[(1-2\psi^{(1)}-\psi^{(2)})\delta_{ij}+\left(\hat{\chi}_{ij}^{(1)}+\frac{1}{2}\hat{\chi}_{ij}^{(2)}\right)\right]\,. \end{aligned} \tag{1}$$

The standard splitting of the perturbations into scalar, transverse (*i.e* divergence-free) vector parts, and transverse trace-free tensor parts with respect to the 3-dimensional space with metric δ_{ij}, can be performed in the following way:

$$\hat{\omega}_i^{(r)} = \partial_i \omega^{(r)} + \omega_i^{(r)}\,, \tag{2}$$

$$\hat{\chi}_{ij}^{(r)} = D_{ij}\chi^{(r)} + \partial_i \chi_j^{(r)} + \partial_j \chi_i^{(r)} + \chi_{ij}^{(r)}\,, \tag{3}$$

where $(r)=(1),(2)$ stand for the order of the perturbations, ω_i and χ_i are transverse vectors ($\partial^i\omega_i^{(r)}=\partial^i\chi_i^{(r)}=0$), $\chi_{ij}^{(r)}$ is a symmetric transverse and trace-free tensor ($\partial^i\chi_{ij}^{(r)}=0$, $\chi^{i(r)}_{\ i}=0$) and $D_{ij}=\partial_i\partial_j-(1/3)\,\delta_{ij}\,\partial^k\partial_k$ is a trace-free operator [3]. Here and in the following latin indices are raised and lowered using δ^{ij} and δ_{ij}, respectively.
For our purposes the metric in Eq. (1) can be simplified. In fact, first-order vector perturbations are zero in the

[3]Notice that our notation is different from that of Refs. [8,9] for the presence of D_{ij}, while it is closer to the one used in Refs. [10,11]. As far as the first-order perturbations are concerned, the metric perturbations ψ and E of Refs. [8,9] are given in our notation as $\psi=\psi^{(1)}+(1/6)\,\partial_i\,\partial^i\chi^{(1)}$ and $E=\chi^{(1)}/2$, respectively. However, no difference appears in the calculations when using the generalized longitudinal gauge in Eq. (28).

presence of a scalar field; moreover, the tensor part gives a negligible contribution to the bispectrum. Thus, in the following we can neglect $\omega_i^{(1)}$, $\chi_i^{(1)}$ and $\chi_{ij}^{(1)}$. However the same is not true for the second order perturbations. In the second-order theory the second-order vector and tensor contributions can be generated by the first-order scalar perturbations even if they are initially zero [7]. Thus we have to take them into account and we shall use the metric

$$
\begin{aligned}
g_{00} &= -a^2(\tau)\left(1+2\phi^{(1)}+\phi^{(2)}\right), \\
g_{0i} &= a^2(\tau)\left(\partial_i\omega^{(1)}+\frac{1}{2}\,\partial_i\omega^{(2)}+\frac{1}{2}\,\omega_i^{(2)}\right), \\
g_{ij} &= a^2(\tau)\left[\left(1-2\psi^{(1)}-\psi^{(2)}\right)\delta_{ij}+D_{ij}\left(\chi^{(1)}+\frac{1}{2}\chi^{(2)}\right)+\frac{1}{2}\left(\partial_i\chi_j^{(2)}+\partial_j\chi_i^{(2)}+\chi_{ij}^{(2)}\right)\right].
\end{aligned} \tag{4}
$$

The controvariant metric tensor is obtained by requiring (up to second order) that $g_{\mu\nu}g^{\nu\lambda}=\delta_\mu{}^\lambda$ and it is given by

$$
\begin{aligned}
g^{00} &= -a^{-2}(\tau)\left(1-2\phi^{(1)}-\phi^{(2)}+4\left(\phi^{(1)}\right)^2-\partial^i\omega^{(1)}\partial_i\omega^{(1)}\right), \\
g^{0i} &= a^{-2}(\tau)\left[\partial^i\omega^{(1)}+\frac{1}{2}\left(\partial^i\omega^{(2)}+\omega^{i(2)}\right)+2\left(\psi^{(1)}-\phi^{(1)}\right)\partial^i\omega^{(1)}-\partial^i\omega^{(1)}D^i{}_k\chi^{(1)}\right], \\
g^{ij} &= a^{-2}(\tau)\left[\left(1+2\psi^{(1)}+\psi^{(2)}+4\left(\psi^{(1)}\right)^2\right)\delta^{ij}-D^{ij}\left(\chi^{(1)}+\frac{1}{2}\chi^{(2)}\right)\right. \\
&\quad \left.-\frac{1}{2}\left(\partial^i\chi^{j(2)}+\partial^j\chi^{i(2)}+\chi^{ij(2)}\right)-\partial^i\omega^{(1)}\partial^j\omega^{(1)}-4\psi^{(1)}D^{ij}\chi^{(1)}+D^{ik}\chi^{(1)}D^j{}_k\chi^{(1)}\right].
\end{aligned} \tag{5}
$$

Using $g_{\mu\nu}$ and $g^{\mu\nu}$ one can calculate the connection coefficients and the Einstein tensor components up to second order in the metric fluctuations. Their complete expressions are contained in Appendix A.

B. Energy-momentum tensor of a scalar field

We shall consider a scalar field $\varphi(\tau,x^i)$ minimally coupled to gravity, whose energy-momentum tensor is given by

$$
T_{\mu\nu} \;=\; \partial_\mu\varphi\,\partial_\nu\varphi \;-\; g_{\mu\nu}\left(\frac{1}{2}\,g^{\alpha\beta}\,\partial_\alpha\varphi\,\partial_\beta\varphi \;+\; V(\varphi)\right), \tag{6}
$$

where $V(\varphi)$ is the potential of the scalar field. A successfull period of inflation can be attained when the potential $V(\varphi)$ is flat enough.

The scalar field can be split into a homogeneous background $\varphi_0(\tau)$ and a perturbation $\delta\varphi(\tau,x^i)$ as

$$
\varphi(\tau,x^i)=\varphi_0(\tau)+\delta\varphi(\tau,x^i)=\varphi_0(\tau)+\delta^{(1)}\varphi(\tau,x^i)+\frac{1}{2}\delta^{(2)}\varphi(\tau,x^i)\,, \tag{7}
$$

where the perturbation has been expanded into a first and a second-order part, respectively. Using the expression (7) into Eq. (6) and calculating $T^\mu_{\ \nu}=g^{\mu\alpha}T_{\alpha\nu}$ up to second order we find

$$
T^\mu_{\ \nu}=T^{\mu(0)}_{\ \nu}+\delta^{(1)}T^\mu_{\ \nu}+\frac{1}{2}\delta^{(2)}T^\mu_{\ \nu}\,, \tag{8}
$$

where $T^{\mu(0)}_{\ \nu}$ corresponds to the background value, and

$$
T^{0(0)}_{\ 0}+\delta^{(1)}T^0_{\ 0}=-\frac{1}{2}a^{-2}{\varphi_0'}^2-V_0+a^{-2}\varphi_0'\left(\phi^{(1)}\ \varphi_0'-\delta^{(1)}\varphi'\right)-\frac{\partial V}{\partial\varphi}\delta^{(1)}\varphi\,, \tag{9}
$$

$$
\begin{aligned}
\delta^{(2)}T^0_{\ 0} &= \frac{2}{a^2}\left[-\frac{1}{2}\delta^{(2)}\varphi'\,\varphi_0' \,-\, \frac{1}{2}\delta^{(2)}\varphi\,\frac{\partial V}{\partial\varphi}\,a^2 \,+\, \frac{1}{2}\,\phi^{(2)}\,{\varphi_0'}^2 \,-\, \frac{1}{2}\left(\delta^{(1)}\varphi'\right)^2 \,-\, \frac{1}{2}\partial^k\delta^{(1)}\varphi\,\partial_k\,\delta^{(1)}\varphi\right. \\
&\quad \left.-\,\frac{1}{2}\left(\delta^{(1)}\varphi\right)^2\frac{\partial^2 V}{\partial\varphi^2}\,a^2 \,-\, 2\left(\phi^{(1)}\right)^2{\varphi_0'}^2 \,+\, 2\,\phi^{(1)}\,\delta^{(1)}\varphi'\,\varphi_0' \,+\, \frac{1}{2}\,\partial^k\omega^{(1)}\,\partial_k\omega^{(1)}\,{\varphi_0'}^2\right],
\end{aligned} \tag{10}
$$

$$T^{0(0)}_{\ i} + \delta^{(1)} T^0_{\ i} = -a^{-2} \left(\varphi_0' \, \partial_i \delta^{(1)} \varphi \right) , \tag{11}$$

$$\delta^{(2)} T^0_{\ i} = \frac{2}{a^2} \left(-\frac{1}{2} \varphi_0' \, \partial_i \delta^{(2)} \varphi - \, \partial_i \, \delta^{(1)} \varphi \, \delta^{(1)} \varphi' \, + \, 2 \, \varphi_0' \, \phi^{(1)} \, \partial_i \delta^{(1)} \varphi \right) , \tag{12}$$

$$T^{i(0)}_{\ 0} + \delta^{(1)} T^i_{\ 0} = {\varphi_0'}^2 \partial^i \omega^{(1)} + \varphi_0' \, \partial^i \delta^{(1)} \varphi \, , \tag{13}$$

$$\begin{aligned} \delta^{(2)} T^i_{\ 0} = \frac{2}{a^2} \Big[& \frac{1}{2} \varphi_0' \, \partial^i \, \delta^{(2)} \varphi \, + \, \frac{1}{2} \left(\partial^i \omega^{(2)} + \omega^{i(2)} \right) {\varphi_0'}^2 \, + \, \delta^{(1)} \varphi' \partial^i \, \delta^{(1)} \varphi \, + \, 2 \, \varphi_0' \, \psi^{(1)} \, \partial^i \, \delta^{(1)} \varphi \\ & - \, 2 \, {\varphi_0'}^2 \, \phi^{(1)} \, \partial^i \omega^{(1)} \, + \, 2 \, \partial^i \omega^{(1)} \, \delta^{(1)} \varphi' \, \varphi_0' \, + \, 2 \, {\varphi_0'}^2 \, \psi^{(1)} \, \partial^i \omega^{(1)} \, - \, \varphi_0' \, D^{ij} \chi^{(1)} \, \partial_j \, \delta^{(1)} \varphi \\ & - {\varphi_0'}^2 \, D^{ij} \chi^{(1)} \, \partial_j \omega^{(1)} \Big] , \end{aligned} \tag{14}$$

$$T^{i(0)}_{\ j} + \delta^{(1)} T^i_{\ j} = \left[\frac{1}{2} a^{-2} {\varphi_0'}^2 - V_0 - \frac{\partial V}{\partial \varphi} \delta^{(1)} \varphi + a^{-2} \varphi_0' \left(\delta^{(1)} \varphi' - \phi^{(1)} \, \varphi_0' \right) \right] \, \delta^i_{\ j} \, , \tag{15}$$

$$\begin{aligned} \delta^{(2)} T^i_{\ j} = \frac{2}{a^2} \Bigg[& \Bigg(\frac{1}{2} \delta^{(2)} \varphi' \, \varphi_0' \, - \, \frac{1}{2} \delta^{(2)} \varphi \, \frac{\partial V}{\partial \varphi} \, a^2 \, - \, \frac{1}{2} \, \phi^{(2)} \, {\varphi_0'}^2 \, + \, \frac{1}{2} \, \left(\delta^{(1)} \varphi' \right)^2 \, - \, \frac{1}{2} \, \partial_k \, \delta^{(1)} \varphi \, \partial^k \delta^{(1)} \varphi \\ & + \, 2 \, \left(\phi^{(1)} \right)^2 \, {\varphi_0'}^2 \, - \, \frac{1}{2} \, \left(\delta^{(1)} \varphi \right)^2 \, \frac{\partial^2 V}{\partial \varphi^2} \, a^2 \, - \, 2 \, \phi^{(1)} \, \delta^{(1)} \varphi' \, \varphi_0' \, - \, \partial^k \omega^{(1)} \, \partial_k \delta^{(1)} \varphi \, \varphi_0' \\ & - \, \frac{1}{2} \, \partial^k \omega^{(1)} \, \partial_k \omega^{(1)} \, {\varphi_0'}^2 \Bigg) \, \delta^i_{\ j} \, + \, \partial^i \delta^{(1)} \varphi \, \partial_j \delta^{(1)} \varphi \, + \varphi_0' \, \partial^i \omega^{(1)} \, \partial_j \, \delta^{(1)} \varphi \Bigg] , \end{aligned} \tag{16}$$

with $V_0 = V(\varphi_0)$. A comment is in order here. As it can be seen from Eq. (5) and Eqs. (10),(12),(14) and (16) the second-order perturbations always contain two different contributions, quantities which are intrinsically of second order, and quantities which are given by the product of two first-order perturbations. As a consequence, when considering the Einstein equations to second order in Section IV, first-order perturbations behave as a source for the intrinsically second-order fluctuations. This is an important issue which was pointed out in different works on second-order perturbation theory [12,13,7] and it plays a central role in deriving our main results on the primordial non-Gaussianity.

III. THE SECOND-ORDER GAUGE-INVARIANT COMOVING CURVATURE PERTURBATION

Let us now focus on the primordial cosmological perturbations produced during a period of inflation driven by the inflaton φ. The fluctuations of the inflaton produce an adiabatic density perturbation which is associated with a perturbation of the spatial curvature ψ. The density/curvature perturbation can be defined in a gauge-invariant manner as the curvature perturbation $\mathcal{R}$ on slices orthogonal to comoving wordlines. At first order, the comoving curvature perturbation is given by the gauge-invariant formula [8]

$$\mathcal{R}^{(1)} = \psi^{(1)} + \frac{\mathcal{H}}{\varphi_0'} \, \delta^{(1)} \varphi \, , \tag{17}$$

where $\mathcal{H} = a'/a$ is the Hubble rate, $\psi^{(1)}$ is the gauge-dependent first-order curvature perturbation and $\delta^{(1)}\varphi$ is the first-order inflaton perturbation in that gauge. An important feature of $\mathcal{R}^{(1)}$ is that it remains constant on super-Hubble scales while approaching the horizon entry.

We want to find the gauge-invariant comoving curvature perturbation $\mathcal{R}$ up to second order. Thus we expand the curvature perturbation ψ and the inflaton perturbation $\delta\varphi$ as $\psi = \psi^{(1)} + \frac{1}{2}\,\psi^{(2)}$ and $\delta\varphi = \delta^{(1)}\varphi + \frac{1}{2}\,\delta^{(2)}\varphi$, according to Eqs. (1) and (7). An infinitesimal coordinate change up to second order induces a gauge transformation of the metric and the scalar field perturbations [6,7]. In particular for a second-order shift of the time coordinate

$$\tau \to \tau - \xi^0_{(1)} + \frac{1}{2} \left(\xi^{0'}_{(1)} \xi^0_{(1)} - \xi^0_{(2)} \right) \tag{18}$$

$\psi^{(2)}$ and $\delta^{(2)}\varphi$ will transform as [6,7]

$$\widetilde{\psi^{(2)}} = \psi^{(2)} + 2\xi^0_{(1)}\left(\psi^{(1)\prime} + 2\mathcal{H}\psi^{(1)}\right) - \left(\mathcal{H}' + 2\mathcal{H}^2\right)\left(\xi^0_{(1)}\right)^2 - \mathcal{H}\xi^0_{(1)}{}'\xi^0_{(1)} - \mathcal{H}\xi^0_{(2)} - \frac{1}{3}\left(2\partial^i\omega^{(1)} - \partial^i\xi^0_{(1)}\right)\partial_i\xi^0_{(1)}\,, \quad (19)$$

$$\widetilde{\delta^{(2)}\varphi} = \delta^{(2)}\varphi + \xi^0_{(1)}\left(\varphi_0''\xi^0_{(1)} + \varphi_0'\xi^0_{(1)}{}' + 2\delta^{(1)}\varphi'\right) + \varphi_0'\xi^0_{(2)}\,. \quad (20)$$

We find that the gauge-invariant comoving curvature perturbation $\mathcal{R} = \mathcal{R}^{(1)} + \frac{1}{2}\mathcal{R}^{(2)}$ is provided by

$$\mathcal{R} = \mathcal{R}^{(1)} + \frac{1}{2}\left[\mathcal{H}\frac{\delta^{(2)}\varphi}{\varphi_0'} + \psi^{(2)}\right] + \frac{1}{2}\frac{\left(\psi^{(1)\prime} + 2\mathcal{H}\psi^{(1)} + \mathcal{H}\delta^{(1)}\varphi'/\varphi_0'\right)^2}{\mathcal{H}' + 2\mathcal{H}^2 - \mathcal{H}\,\varphi_0''/\varphi_0'} - \frac{1}{6}\partial_i\omega^{(1)}\partial^i\omega^{(1)}\,. \quad (21)$$

We devote the rest of this section to illustrate how to find such a quantity.

Let us consider the first two terms in Eq. (21) and how they change under the time coordinate shift of Eq. (18). According to our perturbative expansion

$$\psi + \mathcal{H}\frac{\delta\varphi}{\varphi_0'} = \mathcal{R}^{(1)} + \frac{1}{2}\left[\mathcal{H}\frac{\delta^{(2)}\varphi}{\varphi_0'} + \psi^{(2)}\right]\,, \quad (22)$$

and using Eqs. (19) and (20) this quantity transforms as

$$\widetilde{\psi} + \mathcal{H}\frac{\widetilde{\delta\varphi}}{\varphi_0'} = \psi + \mathcal{H}\frac{\delta\varphi}{\varphi_0'} + \xi^0_{(1)}T - \frac{1}{2}\left(\xi^0_{(1)}\right)^2\left[\mathcal{H}' + 2\mathcal{H}^2 - \frac{\mathcal{H}}{\varphi_0'}\varphi_0''\right] - \frac{1}{6}\left(2\partial^i\omega^{(1)} - \partial^i\xi^0_{(1)}\right)\partial_i\xi^0_{(1)}\,, \quad (23)$$

where we have set $T = \psi^{(1)\prime} + 2\mathcal{H}\psi^{(1)} + \mathcal{H}\delta^{(1)}\varphi'/\varphi_0'$. Notice that

$$\widetilde{T} = T + \xi^0_{(1)}\left[-(\mathcal{H}' + 2\mathcal{H}^2) + \frac{\mathcal{H}}{\varphi_0'}\varphi_0''\right]\,, \quad (24)$$

since the usual first-order transformations for $\psi^{(1)}$ and $\delta^{(1)}\varphi$ are $\widetilde{\psi^{(1)}} = \psi^{(1)} - \mathcal{H}\,\xi^0_{(1)}$ and $\widetilde{\delta^{(1)}\varphi} = \delta^{(1)}\varphi + \varphi_0'\,\xi^0_{(1)}$, respectively. Thus Eq. (23) can be written as

$$\widetilde{\psi} + \mathcal{H}\frac{\widetilde{\delta\varphi}}{\varphi_0'} = \psi + \mathcal{H}\frac{\delta\varphi}{\varphi_0'} + \frac{1}{2}\left(T + \widetilde{T}\right)\xi^0_{(1)} - \frac{1}{6}\left(2\partial^i\omega^{(1)} - \partial^i\xi^0_{(1)}\right)\partial_i\xi^0_{(1)}\,. \quad (25)$$

Solving Eq. (24) for $\xi^0_{(1)}$ and using the the first-order transformation $\widetilde{\omega^{(1)}} = \omega^{(1)} - \xi^0_{(1)}$ to express $\partial_i\xi^0_{(1)}$ and $\partial^i\xi^0_{(1)}$ we finally find

$$\widetilde{\psi} + \mathcal{H}\frac{\widetilde{\delta\varphi}}{\varphi_0'} + \frac{1}{2}\frac{\tilde{T}^2}{\mathcal{H}' + 2\mathcal{H}^2 - \mathcal{H}\varphi_0'/\varphi_0''} - \frac{1}{6}\partial_i\widetilde{\omega^{(1)}}\partial^i\widetilde{\omega^{(1)}} = \psi + \mathcal{H}\frac{\delta\varphi}{\varphi_0'} + \frac{1}{2}\frac{T^2}{\mathcal{H}' + 2\mathcal{H}^2 - \mathcal{H}\varphi_0'/\varphi_0''} - \frac{1}{6}\partial_i\omega^{(1)}\partial^i\omega^{(1)}\,, \quad (26)$$

which shows that $\mathcal{R}$ – the combination on the r.h.s. of this equation – is indeed gauge-independent.

Notice that, by replacing φ with the energy density ρ, we can find in an analogous way a gauge-invariant expression for the curvature perturbation on uniform-density hypersurfaces ζ, which to first order is given by [14] $-\zeta^{(1)} = \psi^{(1)} + \mathcal{H}\delta^{(1)}\rho/\rho_0'$, where ρ_0 is the background energy density. Thus at second order $\zeta = \zeta^{(1)} + \frac{1}{2}\zeta^{(2)}$ is

$$-\zeta = -\zeta^{(1)} + \frac{1}{2}\left[\mathcal{H}\frac{\delta^{(2)}\rho}{\rho_0'} + \psi^{(2)}\right] + \frac{1}{2}\frac{\left(\psi^{(1)\prime} + 2\mathcal{H}\psi^{(1)} + \mathcal{H}\delta^{(1)}\rho'/\rho_0'\right)^2}{\mathcal{H}' + 2\mathcal{H}^2 - \mathcal{H}\,\rho_0''/\rho_0'} - \frac{1}{6}\partial_i\omega^{(1)}\partial^i\omega^{(1)}\,. \quad (27)$$

IV. EINSTEIN EQUATIONS

In this section we shall derive the behaviour on large scales of the comoving curvature perturbation at second order $\mathcal{R}^{(2)}$ introduced in Eq. (21). Our starting point is to calculate the perturbed Einstein equations $\delta^{(2)}G^\mu{}_\nu = (\kappa^2/2)\,\delta^{(2)}T^\mu{}_\nu$ in a generalized longitudinal gauge. Here $\kappa^2 = 8\pi\,G_{\rm N}$ and $T^\mu{}_\nu$ is the energy-momentum tensor of the inflaton field. From the Einstein equations in this gauge we shall pick up an equation for a single unknown function – the potential $\phi^{(2)}$ – in a way similar to the procedure used in Ref. [8] to isolate the equation of motion for $\phi^{(1)}$ in the longitudinal gauge [4]. As in Ref. [8], but at the second-order level in the perturbations, an equation linking $\delta^{(2)}\varphi$ and $\phi^{(2)}$ holds, so that it is possible to obtain an explicit expression for $\delta^{(2)}\varphi$ and hence for the curvature $\mathcal{R}^{(2)}$, once the equation for $\phi^{(2)}$ has been solved. Indeed there are many differences with respect to the first-order case, as it will be evident to the reader from the details of the calculations which follow. The main difficulties arise due to the fact that – contrary to the first-order perturbation theory – also vector and tensor contributions are present, and to the fact that the two scalar potential $\phi^{(2)}$ and $\psi^{(2)}$ differ even in the longitudinal gauge for the presence of source terms which are quadratic in the first-order perturbations.

A. Einstein Equations in the generalized longitudinal gauge

Up to now we have not choosen any particular gauge. Hereafter we will work in a generalized longitudinal gauge defined as

$$\begin{aligned} g_{00} &= -a^2(\tau)(1+2\phi^{(1)}+\phi^{(2)})\,, \\ g_{0i} &= 0\,, \\ g_{ij} &= a^2(\tau)\left[(1-2\psi^{(1)}-\psi^{(2)})\delta_{ij}+\frac{1}{2}\left(\partial_i\chi^{(2)}_j+\partial_j\chi^{(2)}_i+\chi^{(2)}_{ij}\right)\right]\,. \end{aligned} \tag{28}$$

One can obtain the Einstein equations in this gauge either by using directly the metric tensor in Eq. (28) or by using the more general metric of Eq. (4), where no gauge choice has been specified yet, and reduce the equations to the longitudinal gauge only at the end. We have performed both the computations to have a cross check for the equations obtained. In Appendix A we give the expression for the Einstein tensor in the more general form using the metric (4).
We shall now give the Einstein equations in the longitudinal gauge at first and second order in the perturbations, respectively

$$\delta^{(1)}G^0_{\ 0}=\kappa^2\delta^{(1)}T^0_{\ 0} \quad \text{implies} \tag{29}$$

$$6\left(\frac{a'}{a}\right)^2\phi^{(1)}+6\frac{a'}{a}\psi^{(1)\prime}-2\partial_i\,\partial^i\psi^{(1)}=\kappa^2\left(\phi^{(1)}\,{\varphi_0{}'}^2-\delta^{(1)}\varphi'\,\varphi_0{}'-\delta^{(1)}\varphi\,\frac{\partial V}{\partial\varphi}\,a^2\right) \tag{30}$$

$$\delta^{(1)}G^0_{\ i}=\kappa^2\delta^{(1)}T^0_{\ i} \quad \text{implies} \tag{31}$$

$$-2\,\frac{a'}{a}\partial_i\phi^{(1)}-2\,\partial_i\psi^{(1)\prime}=-\kappa^2\,\varphi_0{}'\,\partial^i\delta^{(1)}\varphi \tag{32}$$

$$\delta^{(1)}G^i_{\ j}=\kappa^2\delta^{(1)}T^i_{\ j} \quad \text{implies} \tag{33}$$

$$\begin{aligned} &\left(2\frac{a'}{a}\phi^{(1)\prime}+4\,\frac{a''}{a}\phi^{(1)}-2\left(\frac{a'}{a}\right)^2\phi^{(1)}+\partial_i\,\partial^i\phi^{(1)}+4\,\frac{a'}{a}\psi^{(1)\prime}+2\,\psi^{(1)\prime\prime}-\partial_i\,\partial^i\psi^{(1)}\right)\delta^i_{\ j} \\ &-\partial^i\partial_j\phi^{(1)}+\partial^i\partial_j\psi^{(1)}=\kappa^2\left(-\phi^{(1)}\,{\varphi_0{}'}^2+\delta^{(1)}\varphi'\,\varphi_0{}'-\delta^{(1)}\varphi\,\frac{\partial V}{\partial\varphi}\,a^2\right)\delta^i_{\ j} \end{aligned} \tag{34}$$

$$\delta^{(2)}G^0_{\ 0}=\frac{\kappa^2}{2}\delta^{(2)}T^0_{\ 0} \quad \text{implies} \tag{35}$$

$$\begin{aligned} &3\,\frac{a'}{a}\psi^{(2)\prime}-\partial_i\,\partial^i\,\psi^{(2)}+\left(\frac{a'}{a}\right)^2\phi^{(2)}+\frac{a''}{a}\,\phi^{(2)}-12\left(\frac{a'}{a}\right)^2\left(\psi^{(1)}\right)^2-3\left(\psi^{(1)\prime}\right)^2 \\ &-8\,\psi^{(1)}\partial_i\partial^i\psi^{(1)}-3\,\partial_i\psi^{(1)}\partial^i\psi^{(1)} \end{aligned} \tag{36}$$

---

[4]We recall that the equation of motion for the potential $\phi^{(1)}$ in the longitudinal gauge is [8]
$\phi^{(1)\prime\prime}-\partial_i\partial^i\phi^{(1)}+2\left(\mathcal{H}-\frac{\varphi_0''}{\varphi_0'}\right)\phi^{(1)\prime}+2\left(\mathcal{H}'-\frac{\varphi_0''}{\varphi_0'}\mathcal{H}\right)\phi^{(1)}=0.$

$$= \kappa^2 \Big(-\frac{1}{2}\,\delta^{(2)}\varphi'\,\varphi_0' - \frac{1}{2}\,\delta^{(2)}\varphi\,\frac{\partial V}{\partial\varphi}\,a^2 - \frac{1}{2}\left(\delta^{(1)}\varphi'\right)^2 - \frac{1}{2}\partial^i\delta^{(1)}\varphi\,\partial_i\,\delta^{(1)}\varphi$$
$$- \frac{1}{2}\left(\delta^{(1)}\varphi\right)^2 \frac{\partial^2 V}{\partial\varphi^2}\,a^2 - 2\left(\psi^{(1)}\right)^2 {\varphi_0'}^2 + 2\,\psi^{(1)}\,\delta^{(1)}\varphi'\,\varphi_0'\Big)\,,$$

$$\delta^{(2)}{G^i}_0 = \frac{\kappa^2}{2}\delta^{(2)}{T^i}_0 \quad \text{implies} \tag{37}$$

$$\partial^i\,\psi^{(2)\prime} + \frac{a'}{a}\,\partial^i\,\phi^{(2)} + \frac{1}{4}\partial_k\partial^k\left(\chi^{i(2)}\right)' + 2\,\psi^{(1)\prime}\partial^i\psi^{(1)} + 8\,\psi^{(1)}\partial^i\psi^{(1)\prime} \tag{38}$$
$$= \kappa^2\Big(\frac{1}{2}\,\varphi_0'\,\partial^i\,\delta^{(2)}\varphi + \partial^i\,\delta^{(1)}\varphi\,\delta^{(1)}\varphi' + 2\,\varphi_0'\,\psi^{(1)}\,\partial^i\,\delta^{(1)}\varphi\Big)\,,$$

$$\delta^{(2)}{G^i}_j = \frac{\kappa^2}{2}\delta^{(2)}{T^i}_j \quad \text{implies} \tag{39}$$

$$\Bigg(\frac{1}{2}\,\partial_k\,\partial^k\phi^{(2)} + \frac{a'}{a}\phi^{(2)\prime} + \frac{a''}{a}\phi^{(2)} + \left(\frac{a'}{a}\right)^2\phi^{(2)} - \frac{1}{2}\,\partial_k\,\partial^k\psi^{(2)} + 2\,\frac{a'}{a}\psi^{(2)\prime} + \psi^{(2)\prime\prime} \tag{40}$$
$$- 8\,\frac{a''}{a}\left(\psi^{(1)}\right)^2 + 4\left(\frac{a'}{a}\right)^2\left(\psi^{(1)}\right)^2 - 8\,\frac{a'}{a}\psi^{(1)}\psi^{(1)\prime} - 3\,\partial_k\psi^{(1)}\,\partial^k\psi^{(1)} - 4\,\psi^{(1)}\,\partial_k\,\partial^k\psi^{(1)}$$
$$- \left(\psi^{(1)\prime}\right)^2\Bigg)\,{\delta^i}_j - \frac{1}{2}\partial^i\,\partial_j\,\phi^{(2)} + \frac{1}{2}\partial^i\partial_j\,\psi^{(2)} + \frac{1}{2}\,\frac{a'}{a}\left(\partial_j\chi^{i(2)\prime} + \partial^i\chi^{(2)\prime}_j + {\chi^{i(2)\prime}}_j\right)$$
$$+ \frac{1}{4}\left(\partial_j\chi^{i(2)\prime\prime} + \partial^i\chi^{(2)\prime\prime}_j + {\chi^{i(2)\prime\prime}}_j\right) - \frac{1}{4}\,\partial_k\,\partial^k{\chi^{i(2)}}_j + 2\,\partial^i\psi^{(1)}\,\partial_j\psi^{(1)} + 4\,\psi^{(1)}\,\partial^i\partial_j\psi^{(1)}$$
$$= \kappa^2\Big(\frac{1}{2}\,\delta^{(2)}\varphi'\,\varphi_0' - \frac{1}{2}\,\delta^{(2)}\varphi\,\frac{\partial V}{\partial\varphi}\,a^2 + \frac{1}{2}\left(\delta^{(1)}\varphi'\right)^2 - \frac{1}{2}\,\partial_k\,\delta^{(1)}\varphi\,\partial^k\delta^{(1)}\varphi + 2\left(\psi^{(1)}\right)^2{\varphi_0'}^2$$
$$- \frac{1}{2}\left(\delta^{(1)}\varphi\right)^2\frac{\partial^2 V}{\partial\varphi^2}\,a^2 - 2\,\psi^{(1)}\,\delta^{(1)}\varphi'\,\varphi_0'\Big)\,{\delta^i}_j + \kappa^2\left(\partial^i\delta^{(1)}\varphi\,\partial_j\delta^{(1)}\varphi\right)\,.$$

In writing the perturbed Einstein equations at second order we have set throughout $\phi^{(1)} = \psi^{(1)}$, since in the longitudinal gauge at first-order the two scalar potentials are equal in the case of a scalar field, and to obtain Eqs. (36) and (40) we have made use of the background relations $a''/a = \mathcal{H}^2 + \mathcal{H}'$ and $(\kappa^2/2)\,{\varphi_0'}^2 = \mathcal{H}^2 - \mathcal{H}'$.
We shall now describe how to isolate the equation for the potential $\phi^{(2)}$. We use the $(0-0)$-component of Einstein equations, the divergence of the $(i-0)$-component and the trace of the $(i-j)$-component, both performed with the background metric δ_{ij}. Notice that the divergence and the trace operations make the vector and the tensor modes disappear from the equations. Thus, we are left with three equations in the three unknown functions, $\phi^{(2)}, \psi^{(2)}$, and $\delta^{(2)}\varphi$.
From the divergence of the $(i-0)$-component of Einstein equations it is possible to recover an expression for $\delta^{(2)}\varphi$

$$\frac{1}{2}\delta^{(2)}\varphi = \frac{\left(\psi^{(2)\prime} + \mathcal{H}\,\phi^{(2)} + \triangle^{-1}\alpha\right)}{\kappa^2\,\varphi_0'} - \frac{\triangle^{-1}\beta}{\varphi_0'}\,, \tag{41}$$

where

$$\alpha = 2{\psi^{(1)}}'\partial_i\,\partial^i\psi^{(1)} + 10\,\partial_i{\psi^{(1)}}'\partial^i\psi^{(1)} + 8\psi^{(1)}\,\partial_i\,\partial^i{\psi^{(1)}}'\,, \tag{42}$$
$$\beta = \partial_i\,\partial^i\delta^{(1)}\varphi\,\delta^{(1)}\varphi' + \partial^i\delta^{(1)}\varphi\,\partial_i\delta^{(1)}\varphi' + 2\,\psi^{(1)}\,\partial_i\,\partial^i\delta^{(1)}\varphi\,\varphi_0' + 2\,\partial_i\psi^{(1)}\,\partial^i\delta^{(1)}\varphi\,\varphi_0'\,, \tag{43}$$

and $\triangle^{-1}$ is the inverse of the Laplacian operator for the three spatial-coordinates. The expression (41) and its derivative with respect to the conformal time τ can be used in the trace of the $(i-j)$ equation to obtain

$$\frac{1}{3}\partial_i\,\partial^i\phi^{(2)} - \frac{1}{3}\partial_i\,\partial^i\psi^{(2)} = 8\,\frac{a''}{a}\left(\psi^{(1)}\right)^2 - 4\left(\frac{a'}{a}\right)^2\left(\psi^{(1)}\right)^2 + 8\frac{a'}{a}\,\psi^{(1)}\psi^{(1)\prime} + \frac{7}{3}\,\partial_i\psi^{(1)}\,\partial^i\psi^{(1)} + \frac{8}{3}\,\psi^{(1)}\,\partial_i\,\partial^i\psi^{(1)}$$
$$+ \left(\psi^{(1)\prime}\right)^2 + \triangle^{-1}\alpha' + 2\,\frac{a'}{a}\,\triangle^{-1}\alpha - \kappa^2\,\triangle^{-1}\beta' - 2\frac{a'}{a}\,\kappa^2\,\triangle^{-1}\beta + \kappa^2\Big[\frac{1}{2}\left(\delta^{(1)}\varphi'\right)^2$$
$$- \frac{1}{6}\partial_i\delta^{(1)}\varphi\,\partial^i\delta^{(1)}\varphi + 2\left(\psi^{(1)}\right)^2{\varphi_0'}^2 - \frac{1}{2}\left(\delta^{(1)}\varphi\right)^2\frac{\partial^2 V}{\partial\varphi^2}\,a^2 - 2\psi^{(1)}\delta^{(1)}\varphi'\varphi_0'\Big]\,. \tag{44}$$

Thus a relation between $\psi^{(2)}$ and $\phi^{(2)}$ follows from Eq. (44)

$$\psi^{(2)} = \phi^{(2)} - \triangle^{-1}\gamma, \tag{45}$$

where γ stands for three times the R.H.S of Eq. (44). Eq. (45) shows that the two scalar potentials $\psi^{(2)}$ and $\phi^{(2)}$ differ for quadratic terms in the first-order perturbations, as anticipated above.
Using Eq. (45) we are now in the position to express the other two unknown functions $\psi^{(2)}$ and $\delta^{(2)}\varphi$ in terms solely of $\phi^{(2)}$. From Eq. (41) we finally obtain

$$\frac{1}{2}\delta^{(2)}\varphi = \frac{\left(\phi^{(2)\prime} + \mathcal{H}\phi^{(2)} + \triangle^{-1}\alpha\right)}{\kappa^2\varphi_0'} - \frac{\triangle^{-1}\beta}{\varphi_0'} - \frac{\triangle^{-1}\gamma'}{\kappa^2\varphi_0'}\,. \tag{46}$$

Plugging Eqs. (45) and (46) into the $(0-0)$ Einstein equation, the equation of motion for $\phi^{(2)}$ is derived

$$\begin{aligned}
&\phi^{(2)\prime\prime} - \partial_i\partial^i\phi^{(2)} + 2\left(\mathcal{H} - \frac{\varphi_0''}{\varphi_0'}\right)\phi^{(2)\prime} + 2\left(\mathcal{H}' - \frac{\varphi_0''}{\varphi_0'}\mathcal{H}\right)\phi^{(2)} = \\
&12\,\mathcal{H}^2\left(\psi^{(1)}\right)^2 + 3\left(\psi^{(1)\prime}\right)^2 + 8\psi^{(1)}\partial_i\partial^i\psi^{(1)} + 3\partial_i\psi^{(1)}\partial^i\psi^{(1)} + 2\left(\mathcal{H} + \frac{\varphi_0''}{\varphi_0'}\right)\triangle^{-1}\alpha - \triangle^{-1}\alpha' \\
&-\,2\kappa^2\left(\mathcal{H} + \frac{{\varphi_0}''}{\varphi_0'}\right)\triangle^{-1}\beta + \kappa^2\triangle^{-1}\beta' - \gamma + \left(\mathcal{H} - 2\frac{\varphi_0''}{\varphi_0'}\right)\triangle^{-1}\gamma' + \triangle^{-1}\gamma'' \\
&+\,\kappa^2\left(-\frac{1}{2}\left(\delta^{(1)}\varphi'\right)^2 - \frac{1}{2}\partial_i\delta^{(1)}\varphi\partial^i\delta^{(1)}\varphi - 2\left(\psi^{(1)}\right)^2\varphi_0'^2 - \frac{1}{2}\left(\delta^{(1)}\varphi\right)^2\frac{\partial^2 V}{\partial\varphi^2}a^2 + 2\psi^{(1)}\delta^{(1)}\varphi'\varphi_0'\right).
\end{aligned} \tag{47}$$

Eq. (47) is our master equation. Before solving it, let us stress two important points. First, no approximation has been made up to now. In particular notice that this equation is exact at any order in the expansion in terms of the slow-roll parameters. Secondly, the L.H.S. of Eq. (47) is exactly the same as in the equation for $\phi^{(1)}$ in the longitudinal gauge at first order (see the footnote 4). However, at second order, the key point is that Eq. (47) for $\phi^{(2)}$ is not homogeneous, but there is a source made up of terms which are quadratic in the first-order perturbations. Notice that, as in the first-order calculation, it is not necessary to use the perturbed Klein-Gordon equation to close the system of the evolution equations for the fluctuactions. Nevertheless, we also calculated the Klein-Gordon equation at second order in the inflaton and metric fluctuactions; this is reported in Appendix B.

B. The large-scale curvature perturbation $\mathcal{R}^{(2)}$ in the slow-roll approximation

We shall now solve Eq. (47) in order to obtain the expression for the comoving curvature perturbation $\mathcal{R}^{(2)}$ defined in Eq. (21). First we rewrite Eq. (47) in cosmic time $dt = a\,d\tau$, since it is more easy in this way to recognize the slow-roll parameters $\epsilon = -\dot{H}/H^2$ and $\eta = \epsilon - (\ddot{\varphi}_0/H\dot{\varphi}_0)$ [1]. During a period of inflation ϵ and η must be $\ll 1$, but only at a certain point we will perform an expansion to lowest-order in the slow-roll parameters. Moreover, where possible, we shall neglect some terms which give a subdominant contribution on large scales.

Using cosmic time, Eq. (47) becomes

$$\begin{aligned}
&\ddot{\phi}^{(2)} + H\left(1 - 2\frac{\ddot{\varphi}_0}{H\dot{\varphi}_0}\right)\dot{\phi}^{(2)} + 2H^2\left(\frac{\dot{H}}{H^2} - \frac{\ddot{\varphi}_0}{H\dot{\varphi}_0}\right)\phi^{(2)} - \frac{1}{a^2}\partial_i\,\partial^i\phi^{(2)} = \\
&-\,24H^2\left(1 + \frac{\dot{H}}{H^2}\right)\left(\psi^{(1)}\right)^2 - 24H\psi^{(1)}\dot{\psi}^{(1)} - \frac{4}{a^2}\partial_i\psi^{(1)}\partial^i\psi^{(1)} - \frac{2}{a}H\left(1 - \frac{\ddot{\varphi}_0}{H\dot{\varphi}_0}\right)\triangle^{-1}\alpha \\
&-\,\frac{4}{a}\triangle^{-1}\dot{\alpha} + 2\frac{\kappa^2}{a}H\left(1 - \frac{\ddot{\varphi}_0}{H\dot{\varphi}_0}\right)\triangle^{-1}\beta + 4\frac{\kappa^2}{a}\triangle^{-1}\dot{\beta} - 2\frac{\ddot{\varphi}_0}{\dot{\varphi}_0}\triangle^{-1}\dot{\gamma} + \triangle^{-1}\ddot{\gamma} \\
&-\,\kappa^2\left(2\left(\delta^{(1)}\dot{\varphi}\right)^2 + 8\dot{\varphi}_0{}^2\left(\psi^{(1)}\right)^2 - \frac{\partial^2 V}{\partial\varphi^2}\left(\delta^{(1)}\varphi\right)^2 - 8\,\dot{\varphi}_0\psi^{(1)}\left(\delta^{(1)}\varphi\right)^{\cdot}\right),
\end{aligned} \tag{48}$$

where we have replaced the term $-\gamma$ appearing in Eq. (47) with its explicit definition given by Eqs. (44) and (45). Notice that in the source on the R.H.S. of Eq. (48) there appears always the combination $\Delta^{-1}\left(\kappa^2\beta/a - \alpha/a\right)$ and its derivative with respect to cosmic time. Let us now calculate such a combination. The quantity α defined in Eq. (42) can be rewritten as

$$\frac{\alpha}{a} = 2\partial_i\,\partial^i\left(\psi^{(1)}\dot{\psi}^{(1)}\right) + 6\left(\partial_i\psi^{(1)}\partial^i\dot{\psi}^{(1)} + \psi^{(1)}\partial_i\,\partial^i\dot{\psi}^{(1)}\right) . \tag{49}$$

Using the equation of motion (30) to express $\left(\delta^{(1)}\varphi\right)^{\cdot}$, the quantity β, Eq. (43), turns out to be

$$\begin{aligned}\frac{\beta}{a} = &\frac{1}{2}\frac{\ddot{\varphi}_0}{\dot{\varphi}_0}\partial_i\,\partial^i\left(\delta^{(1)}\varphi\right)^2 + 3\dot{\varphi}_0\,\partial_i\psi^{(1)}\partial^i\delta^{(1)}\varphi + 3\dot{\varphi}_0\,\psi^{(1)}\partial_i\,\partial^i\delta^{(1)}\varphi\\ &+ \frac{2}{\kappa^2\dot{\varphi}_0}\partial_k\delta^{(1)}\varphi\,\partial^k\left(\partial_i\,\partial^i\psi^{(1)}\right) + \frac{2}{\kappa^2\dot{\varphi}_0}\partial_i\,\partial^i\delta^{(1)}\varphi\,\partial_k\,\partial^k\delta^{(1)}\psi^{(1)} .\end{aligned} \tag{50}$$

Using the equation of motion for $\psi^{(1)}$ in the longitudinal gauge

$$\dot{\psi}^{(1)} + H\psi^{(1)} = \frac{\kappa^2}{2}\dot{\varphi}_0\delta^{(1)}\varphi\,, \tag{51}$$

which can be derived from Eq. (32) with $\phi^{(1)} = \psi^{(1)}$, we get

$$\begin{aligned}\kappa^2\frac{\beta}{a} - \frac{\alpha}{a} = &\frac{\kappa^2}{2}\frac{\ddot{\varphi}_0}{\dot{\varphi}_0}\partial_i\,\partial^i\left(\delta^{(1)}\varphi\right)^2 + 3H\partial_i\,\partial^i\left(\psi^{(1)}\right)^2 - 2\partial_i\,\partial^i\left(\psi^{(1)}\dot{\psi}^{(1)}\right)\\ &+ \frac{2}{a^2\dot{\varphi}_0}\partial_k\delta^{(1)}\varphi\,\partial^k\left(\partial_i\,\partial^i\psi^{(1)}\right) + \frac{2}{a^2\dot{\varphi}_0}\partial_i\,\partial^i\delta^{(1)}\varphi\,\partial_k\,\partial^k\delta^{(1)}\psi^{(1)} ,\end{aligned} \tag{52}$$

and therefore

$$\begin{aligned}\Delta^{-1}\left(\kappa^2\frac{\beta}{a} - \frac{\alpha}{a}\right) = &\frac{\kappa^2}{2}\frac{\ddot{\varphi}_0}{\dot{\varphi}_0}\left(\delta^{(1)}\varphi\right)^2 + 3H\left(\psi^{(1)}\right)^2 - 2\left(\psi^{(1)}\dot{\psi}^{(1)}\right)\\ &+ \frac{2}{\dot{\varphi}_0}\Delta^{-1}\left[\partial_k\delta^{(1)}\varphi\,\partial^k\left(\partial_i\,\partial^i\psi^{(1)}\right) + \partial_i\,\partial^i\delta^{(1)}\varphi\,\partial_k\,\partial^k\psi^{(1)}\right] .\end{aligned} \tag{53}$$

The master equation (48) can be simplified if we drop those terms which are next to leading-order in the slow-roll parameters

$$\ddot{\phi}^{(2)} + H\dot{\phi}^{(2)} - \frac{1}{a^2}\partial_i\,\partial^i\phi^{(2)} = \tag{54}$$

$$\begin{aligned}&- 12\kappa^2 H\delta^{(1)}\varphi\,\psi^{(1)}\dot{\varphi}_0 + 3\kappa^2 H\frac{\ddot{\varphi}_0}{\dot{\varphi}_0}\left(\delta^{(1)}\varphi\right)^2 - 12H\psi^{(1)}\dot{\psi}^{(1)} + 18H^2\left(\psi^{(1)}\right)^2\\ &+ 4\left[\frac{\kappa^2}{2}\frac{\ddot{\varphi}_0}{\dot{\varphi}_0}\left(\delta^{(1)}\varphi\right)^2 + 3H\left(\psi^{(1)}\right)^2 - 2\left(\psi^{(1)}\dot{\psi}^{(1)}\right)\right]^{\cdot} - 2\frac{\ddot{\varphi}_0}{\dot{\varphi}_0}\triangle^{-1}\dot{\gamma} + \triangle^{-1}\ddot{\gamma}\\ &+ \frac{12H}{a^2\dot{\varphi}_0}\Delta^{-1}\left[\partial_k\delta^{(1)}\varphi\,\partial^k\left(\partial_i\,\partial^i\psi^{(1)}\right) + \partial_i\,\partial^i\delta^{(1)}\varphi\,\partial_k\,\partial^k\psi^{(1)}\right]\\ &+ \frac{8}{a^2\dot{\varphi}_0}\Delta^{-1}\left[\partial_k\delta^{(1)}\varphi\,\partial^k\left(\partial_i\,\partial^i\psi^{(1)}\right) + \partial_i\,\partial^i\delta^{(1)}\varphi\,\partial_k\,\partial^k\psi^{(1)}\right]^{\cdot} - \frac{4}{a^2}\partial_i\psi^{(1)}\partial^i\psi^{(1)} + \kappa^2\frac{\partial^2 V}{\partial\varphi^2}\left(\delta^{(1)}\varphi\right)^2\\ &- \frac{8}{\kappa^2\dot{\varphi}_0{}^2}\left(\frac{\partial_i\,\partial^i\psi^{(1)}}{a^2}\right)^2 + \frac{8}{a^2}\psi^{(1)}\partial_i\,\partial^i\psi^{(1)} - \frac{8}{a^2}\frac{\ddot{\varphi}_0}{\dot{\varphi}_0}\frac{\delta^{(1)}\varphi}{\dot{\varphi}_0}\partial_i\,\partial^i\psi^{(1)} ,\end{aligned} \tag{55}$$

where we have used Eq. (51) so that $-24H^2\left(1+\dot{H}/H^2\right)\left(\psi^{(1)}\right)^2 - 24H\psi^{(1)}\dot{\psi}^{(1)} = -24\dot{H}\left(\psi^{(1)}\right)^2 - 12\kappa^2 H\delta^{(1)}\varphi\,\psi^{(1)}\dot{\varphi}_0$, we have used the expression for $\left(\delta^{(1)}\varphi\right)^{\cdot}$ from Eq. (30), and we also explicitly written the combination (53).

Let us notice that through Eq. (51) we find

$$\begin{aligned}&- 12\kappa^2 H\delta^{(1)}\varphi\,\psi^{(1)}\dot{\varphi}_0 + 3\kappa^2 H\frac{\ddot{\varphi}_0}{\dot{\varphi}_0}\left(\delta^{(1)}\varphi\right)^2 - 12H\psi^{(1)}\dot{\psi}^{(1)} + 18H^2\left(\psi^{(1)}\right)^2 =\\ &- 18\kappa^2 H\delta^{(1)}\varphi\,\psi^{(1)}\dot{\varphi}_0 + 3\kappa^2 H\frac{\ddot{\varphi}_0}{\dot{\varphi}_0}\left(\delta^{(1)}\varphi\right)^2 + 30H^2\left(\psi^{(1)}\right)^2 =\\ &- 12H\left(\psi^{(1)\,2}\right)^{\cdot} + 6\left(\dot{\psi}^{(1)}\right)^2 - \kappa^2\frac{\partial^2 V}{\partial\varphi^2}\left(\delta^{(1)}\varphi\right)^2 ,\end{aligned} \tag{56}$$

where the last passage is valid to lowest-order in the slow-roll parameters. Thus we finally obtain

$$
\begin{aligned}
\ddot{\phi}^{(2)}+&H\dot{\phi}^{(2)}+2H^2\left(\frac{\dot{H}}{H^2}-\frac{\ddot{\varphi}_0}{H\dot{\varphi}_0}\right)\phi^{(2)}-\frac{1}{a^2}\partial_i\,\partial^i\phi^{(2)}=\\
&-\ 12H\left(\psi^{(1)\,2}\right)^{\cdot}+6\left(\dot{\psi}^{(1)}\right)^2+4\left[\frac{\kappa^2}{2}\frac{\ddot{\varphi}_0}{\dot{\varphi}_0}\left(\delta^{(1)}\varphi\right)^2+3H\left(\psi^{(1)}\right)^2-2\left(\psi^{(1)}\dot{\psi}^{(1)}\right)\right]^{\cdot}\\
&-\ 2\frac{\ddot{\varphi}_0}{\dot{\varphi}_0}\triangle^{-1}\dot{\gamma}+\triangle^{-1}\ddot{\gamma}+\frac{12H}{a^2\dot{\varphi}_0}\Delta^{-1}\left[\partial_k\delta^{(1)}\varphi\,\partial^k\left(\partial_i\,\partial^i\psi^{(1)}\right)+\partial_i\,\partial^i\delta^{(1)}\varphi\,\partial_k\,\partial^k\psi^{(1)}\right]\\
&+\ \frac{8}{a^2\dot{\varphi}_0}\Delta^{-1}\left[\partial_k\delta^{(1)}\varphi\,\partial^k\left(\partial_i\,\partial^i\psi^{(1)}\right)+\partial_i\,\partial^i\delta^{(1)}\varphi\,\partial_k\,\partial^k\psi^{(1)}\right]^{\cdot}-\frac{4}{a^2}\partial_i\psi^{(1)}\partial^i\psi^{(1)}\\
&-\ \frac{8}{\kappa^2\dot{\varphi}_0{}^2}\left(\frac{\partial_i\,\partial^i\psi^{(1)}}{a^2}\right)^2+\frac{8}{a^2}\psi^{(1)}\partial_i\,\partial^i\psi^{(1)}-\frac{8}{a^2}\frac{\ddot{\varphi}_0}{\dot{\varphi}_0}\frac{\delta^{(1)}\varphi}{\dot{\varphi}_0}\partial_i\,\partial^i\psi^{(1)}\,.
\end{aligned}
\tag{57}
$$

Integrating Eq. (57) we find

$$
\begin{aligned}
\dot{\phi}^{(2)}+H\phi^{(2)}=&-12H\left(\psi^{(1)}\right)^2+4\left[\frac{\kappa^2}{2}\frac{\ddot{\varphi}_0}{\dot{\varphi}_0}\left(\delta^{(1)}\varphi\right)^2+3H\left(\psi^{(1)}\right)^2-2\left(\psi^{(1)}\dot{\psi}^{(1)}\right)\right]-\frac{\ddot{\varphi}_0}{\dot{\varphi}_0}\triangle^{-1}\gamma+\triangle^{-1}\dot{\gamma}\\
&+6\int\left(\dot{\psi}^{(1)}\right)^2dt+\int\frac{12H}{a^2\dot{\varphi}_0}\Delta^{-1}\left[\partial_k\delta^{(1)}\varphi\,\partial^k\left(\partial_i\,\partial^i\psi^{(1)}\right)+\partial_i\,\partial^i\delta^{(1)}\varphi\,\partial_k\,\partial^k\psi^{(1)}\right]dt\\
&-4\int\frac{1}{a^2}\left(\partial_i\psi^{(1)}\partial^i\psi^{(1)}\right)dt-\int\frac{8}{\kappa^2\dot{\varphi}_0{}^2}\left(\frac{\partial_i\,\partial^i\psi^{(1)}}{a^2}\right)^2dt+\int\frac{8}{a^2}\psi^{(1)}\partial_i\,\partial^i\psi^{(1)}dt\\
&-\int\frac{8}{a^2}\frac{\ddot{\varphi}_0}{\dot{\varphi}_0}\frac{\delta^{(1)}\varphi}{\dot{\varphi}_0}\partial_i\,\partial^i\psi^{(1)}dt\,.
\end{aligned}
\tag{58}
$$

We are now in the position to calculate the second-order comoving curvature perturbation $\mathcal{R}^{(2)}$. From Eqs. (21), (45) and (46) we get

$$
\begin{aligned}
\mathcal{R}^{(2)}=&\ 2\ \frac{H}{\kappa^2\dot{\varphi}_0{}^2}\left[\dot{\phi}^{(2)}+H\phi^{(2)}+\triangle^{-1}\left(\frac{\alpha}{a}\right)\right]-2\frac{H}{\dot{\varphi}_0{}^2}\triangle^{-1}\left(\frac{\beta}{a}\right)-2\frac{H}{\kappa^2\dot{\varphi}_0{}^2}\triangle^{-1}\dot{\gamma}+\phi^{(2)}-\triangle^{-1}\gamma\\
&+\frac{1}{2}\frac{\left(\dot{\psi^{(1)}}+2H\psi^{(1)}+H\delta^{(1)}\dot{\varphi}/\dot{\varphi}_0\right)^2}{H^2\left(2+\dot{H}/H^2-\ddot{\varphi}_0/H\dot{\varphi}_0\right)}\,,
\end{aligned}
\tag{59}
$$

where the last term is the part of $\mathcal{R}^{(2)}$ in Eq. (21) which is quadratic in the first-order perturbations. Note that it is of the order of $\mathcal{O}(\epsilon^2,\eta^2)$ and thus we shall neglect it in the following. In Eq. (59) there appears once more the combination $\Delta^{-1}\left(\kappa^2\beta/a-\alpha/a\right)$. Thus using Eq. (53) and Eq. (58) we obtain

$$
\begin{aligned}
\mathcal{R}^{(2)}=&\ 3H\frac{\ddot{\varphi}_0}{\dot{\varphi}_0}\left(\frac{\delta^{(1)}\varphi}{\dot{\varphi}_0}\right)^2-12\frac{H}{\kappa^2\dot{\varphi}_0{}^2}\psi^{(1)}\dot{\psi}^{(1)}-6\frac{H^2}{\kappa^2\dot{\varphi}_0{}^2}\left(\psi^{(1)}\right)^2+\frac{12H}{\kappa^2\dot{\varphi}_0{}^2}\int\left(\dot{\psi}^{(1)}\right)^2dt\\
&+\frac{24H^2}{\kappa^2\dot{\varphi}_0{}^3}\int\frac{1}{a^2}\Delta^{-1}\left[\partial_k\delta^{(1)}\varphi\,\partial^k\left(\partial_i\,\partial^i\psi^{(1)}\right)+\partial_i\,\partial^i\delta^{(1)}\varphi\,\partial_k\,\partial^k\psi^{(1)}\right]dt-\frac{8H}{\kappa^2\dot{\varphi}_0{}^2}\int\frac{1}{a^2}\left(\partial_i\psi^{(1)}\partial^i\psi^{(1)}\right)dt\\
&-\frac{16H}{\kappa^4\dot{\varphi}_0{}^4}\int\left(\frac{\partial_i\,\partial^i\psi^{(1)}}{a^2}\right)^2dt+\frac{16H}{\kappa^2\dot{\varphi}_0{}^2}\int\frac{1}{a^2}\psi^{(1)}\partial_i\,\partial^i\psi^{(1)}dt-\frac{16H}{\kappa^2\dot{\varphi}_0{}^3}\frac{\ddot{\varphi}_0}{\dot{\varphi}_0}\int\frac{1}{a^2}\delta^{(1)}\varphi\,\partial_i\,\partial^i\psi^{(1)}dt\\
=&-3\eta H^2\left(\frac{\delta^{(1)}\varphi}{\dot{\varphi}_0}\right)^2+6\frac{H^2}{\kappa^2\dot{\varphi}_0{}^2}\left(\psi^{(1)}\right)^2-6H\frac{\delta\varphi}{\dot{\varphi}_0}\psi^{(1)}+3\epsilon H^2\frac{(\delta\varphi)^2}{\dot{\varphi}_0{}^2}+\frac{12H}{\kappa^2\dot{\varphi}_0{}^2}\int\left(\dot{\psi}\right)^2dt\\
&+\frac{24H^2}{\dot{\varphi}_0{}^3}\int\frac{1}{a^2}\Delta^{-1}\left[\partial_k\delta^{(1)}\varphi\,\partial^k\left(\partial_i\,\partial^i\psi^{(1)}\right)+\partial_i\,\partial^i\delta^{(1)}\varphi\,\partial_i\,\partial^i\psi^{(1)}\right]dt-\frac{8H}{\kappa^2\dot{\varphi}_0{}^2}\int\frac{1}{a^2}\left(\partial_i\psi^{(1)}\partial^i\psi^{(1)}\right)dt\\
&-\frac{16H}{\kappa^4\dot{\varphi}_0{}^4}\int\left(\frac{\partial_i\,\partial^i\psi^{(1)}}{a^2}\right)^2dt+\frac{16H}{\kappa^2\dot{\varphi}_0{}^2}\int\frac{1}{a^2}\psi^{(1)}\partial_i\,\partial^i\psi^{(1)}dt-\frac{16H}{\kappa^2\dot{\varphi}_0{}^3}\frac{\ddot{\varphi}_0}{\dot{\varphi}_0}\int\frac{1}{a^2}\delta^{(1)}\varphi\,\partial_i\,\partial^i\psi^{(1)}dt\\
&-2\frac{H}{\kappa^2\dot{\varphi}_0{}^2}\frac{\ddot{\varphi}_0}{\dot{\varphi}_0}\triangle^{-1}\gamma-\triangle^{-1}\gamma\,,
\end{aligned}
\tag{60}
$$

where the last passage is valid to lowest order in the slow-roll parameters and we have used the equation of motion (51). In Eq. (60) the terms which are not integrated in time can be safely treated in the long-wavelength limit. To lowest order in the slow-roll parameters and on large scales, $k \ll aH$, $\psi^{(1)}$ can be considered as constant and

$$\psi^{(1)} = \frac{\kappa^2}{2}\frac{\dot{\varphi}_0}{H}\delta^{(1)}\varphi = \epsilon\, H\frac{\delta^{(1)}\varphi}{\dot{\varphi}_0} \tag{61}$$

which can be derived from the equations of motion (51) and (B.4) in the longitudinal gauge for $\psi^{(1)}$ and $\delta^{(1)}\varphi$, respectively. On the other hand, from the definition of the comoving curvature perturbation at first order, Eq. (17), and using Eq. (61) we can write $\mathcal{R}^{(1)} = H\delta^{(1)}\varphi/\dot{\varphi}_0$ to lowest order in the slow-roll parameters and on large scales. Thus in these approximations Eq. (61) can be rewritten as

$$\psi^{(1)} = \epsilon\,\mathcal{R}^{(1)}\,. \tag{62}$$

Performing various integrations by parts in expression (60), using the perturbation equations at first-order and properly substracting the contributions in the far ultraviolet, we arrive at the final expression for the comoving curvature perturbation

$$\mathcal{R}^{(2)} = (\eta - 3\epsilon)\left(\mathcal{R}^{(1)}\right)^2 + \mathcal{I}\,, \tag{63}$$

where

$$\begin{aligned}\mathcal{I} = &-\frac{2}{\epsilon}\int \frac{1}{a^2}\psi^{(1)}\partial_i\,\partial^i\psi^{(1)}dt - \frac{4}{\epsilon}\int\frac{1}{a^2}\left(\partial_i\psi^{(1)}\partial^i\psi^{(1)}\right)dt\\ &-\frac{4}{\epsilon}\int\left(\ddot{\psi}^{(1)}\right)^2 dt + (\epsilon-\eta)\,\triangle^{-1}\partial_i R^{(1)}\partial^i R^{(1)}\,.\end{aligned} \tag{64}$$

V. DISCUSSION AND CONCLUSIONS

In this paper we have provided a complete analysis of the second-order scalar perturbations during inflation leading to the derivation of the gauge-invariant comoving curvature perturbation $\mathcal{R}$.

The comoving curvature perturbation receives a contribution which is quadratic in $\mathcal{R}^{(1)}$. The total curvature perturbation will then have a non-Gaussian (χ^2) component. Reminding that the gauge-invariant potential Φ (which is related to Bardeen's variable [10] by $\Phi = -\Phi_H$) and the curvature perturbation $\mathcal{R}$ are related by $\Phi = \frac{3}{5}\mathcal{R}$, the following simple relation in configuration space holds

$$\Phi(\mathbf{x}) = \Phi_{\rm Gauss}(\mathbf{x}) + \int d^3y\, d^3z\,\mathcal{K}\left(\mathbf{y},\mathbf{z}\right)\Phi_{\rm Gauss}(\mathbf{x}-\mathbf{y})\Phi_{\rm Gauss}(\mathbf{x}-\mathbf{z}) + \text{constant} \tag{65}$$

which is valid on superhorizon scales and where the constant is such that $\langle\Phi(\mathbf{x})\rangle = 0$. Here $\Phi_{\rm Gauss} = \frac{3}{5}\mathcal{R}^{(1)}$ is a Gaussian random field. The non-Gaussianity kernel in momentum space is given by

$$\mathcal{K}\left(\mathbf{k}_1,\mathbf{k}_2\right) = \frac{5}{6}\left(\eta - 3\epsilon\right) + f_{\mathcal{K}}\left(\mathbf{k}_1,\mathbf{k}_2\right)\,, \tag{66}$$

where $f_{\mathcal{K}}\left(\mathbf{k}_1,\mathbf{k}_2\right)$ is directly related to the function $\mathcal{I}$ and is first-order in the slow-roll parameters. The gravitational potential bispectrum then reads

$$\langle\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)\Phi(\mathbf{k}_3)\rangle = (2\pi)^3\,\delta^{(3)}\left(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3\right)\left[2\,\mathcal{K}\left(\mathbf{k}_1,\mathbf{k}_2\right)\mathcal{P}_\Phi(\mathbf{k}_1)\mathcal{P}_\Phi(\mathbf{k}_2) + \text{cyclic}\right]\,, \tag{67}$$

where $\mathcal{P}_\Phi(\mathbf{k})$ is the power spectrum of the gravitational potential. We can also define an effective "momentum-dependent" parameter $f_{\rm NL}$ given by[5]

[5]Notice that according to our definition of Φ, our $f_{\rm NL}$ has the opposite sign of that in Ref. [15].

$$
f_{\mathrm{NL}} \sim \frac{5}{12}\left(n_S-1\right)+f_{\mathcal{K}}\left(\mathbf{k}_1, \mathbf{k}_2\right) , \tag{68}
$$

where we have made use of the relation between the spectral index $n_S = 1 - 6\epsilon + 2\eta$ for the scalar perturbations and the slow-roll parameters. In the final expression (68) we have not written possible terms of the order of $\mathcal{O}(\epsilon^2, \eta^2)\Delta N$ which might be sizeable for certain classes of inflationary models (ΔN is the number of e-foldings from the time at which a given scale crosses the horizon and the end of inflation; for large-scale CMB anisotropies $\Delta N \approx 60$). Whether these terms cancel out, or equivalently whether the curvature perturbation remains constant on super-horizon scales also at next-to-leading order in the slow-roll parameters remains an open issue. However, if present, terms of order of $\mathcal{O}(\epsilon^2, \eta^2)\Delta N$ resemble those found in Refs. [2,16], coming from the self-interactions of the inflaton field. The primordial gauge-invariant potential bispectrum leads to a nonzero CMB bispectrum via the Sachs-Wolfe effect $(\Delta T/T)_{SW} = (1/3)\Phi$.

Deviations from a scale-invariant spectrum can make the primordial non-Gaussianity non-negligible. The possible presence of non-Gaussianity in primordial cosmological perturbations is only mildly constrained by existing observations [17,18]. Recent analyses of the angular bispectrum from 4-year COBE data [19] yield a weak upper limit, $|f_{\mathrm{NL}}| < 1.5 \times 10^3$. The analysis of the diagonal angular bispectrum of the Maxima dataset [20] also provides a very weak constraint: $|f_{\mathrm{NL}}| < 2330$. According to Ref. [15], however, the minimum value of $|f_{\mathrm{NL}}|$ that will become detectable from the analysis of MAP and *Planck* data, after properly subtracting detector noise and foreground contamination, is about 20 and 5, respectively. These results imply that detecting non-Gaussianity at the level emerging from our second-order calculation will represent a challenge for the forthcoming satellite experiments.

Note added: While revising this paper, a related work by Maldacena, astro-ph/0210603, appeared, where the three-point correlation function for the curvature perturbation is obtained by computing the cubic term contributions to the Lagrangian. Our results, where comparison is possible, agree with those in astro-ph/0210603.

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APPENDIX A: PERTURBING GRAVITY AT SECOND ORDER

1. Basic notation

The number of spatial dimensions is $n = 3$. Greek indices $(\alpha, \beta, ..., \mu, \nu, ...)$ run from 0 to 3, while latin indices $(a, b, ..., i, j, k, ...m, n...)$ run from 1 to 3. The total spacetime metric $g_{\mu\nu}$ has signature $(-,+,+,+)$. The connection coefficients are defined as

$$
\Gamma^{\alpha}_{\beta\gamma} \,=\, \frac{1}{2}\, g^{\alpha\rho}\left(\frac{\partial g_{\rho\gamma}}{\partial x^{\beta}} \,+\, \frac{\partial g_{\beta\rho}}{\partial x^{\gamma}} \,-\, \frac{\partial g_{\beta\gamma}}{\partial x^{\rho}}\right) . \tag{A.1}
$$

The Riemann tensor is defined as

$$
R^{\alpha}{}_{\beta\mu\nu} = \Gamma^{\alpha}_{\beta\nu,\mu} - \Gamma^{\alpha}_{\beta\mu,\nu} + \Gamma^{\alpha}_{\lambda\mu}\Gamma^{\lambda}_{\beta\nu} - \Gamma^{\alpha}_{\lambda\nu}\Gamma^{\lambda}_{\beta\mu} . \tag{A.2}
$$

The Ricci tensor is a contraction of the Riemann tensor

$$
R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} , \tag{A.3}
$$

and in terms of the connection coefficient it is given by

$$
R_{\mu\nu} \,=\, \partial_\alpha\, \Gamma^{\alpha}_{\mu\nu} \,-\, \partial_\mu\, \Gamma^{\alpha}_{\nu\alpha} \,+\, \Gamma^{\alpha}_{\sigma\alpha}\, \Gamma^{\sigma}_{\mu\nu} \,-\, \Gamma^{\alpha}_{\sigma\nu}\, \Gamma^{\sigma}_{\mu\alpha} . \tag{A.4}
$$

The Ricci scalar is given by contracting the Ricci tensor

$$
R = R^{\mu}{}_{\mu} . \tag{A.5}
$$

The Einstein tensor is defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\,. \tag{A.6}$$

The Einstein equations are written as $G_{\mu\nu} = \kappa^2 T_{\mu\nu}$, so that $\kappa^2 = 8\pi G_{\rm N}$, where $G_{\rm N}$ is the usual Newtonian gravitational constant.
In the following expressions we have chosen a specific ordering of the terms. In the expressions in which two spatial indices appear, such as Eq. (A.18), we have assembled together the terms proportional to δ_{ij}. The intrinsically second-order terms precede the source terms which are quadratic in the first-order perturbations. The second-order fluctuations have been listed in the following order as $\phi^{(2)}$, $\psi^{(2)}$, $\omega^{(2)}$, $\omega_i^{(2)}$, $\chi^{(2)}$, $\chi_i^{(2)}$ and $\chi_{ij}^{(2)}$, respectively. This ordering simplifies the analogy between the first-order and the second-order equations and allows to obtain immediately the expressions in a given gauge.

2. The connection coefficients

In a spatially flat Robertson-Walker background the connection coefficients are

$$\Gamma^0_{00} = \frac{a'}{a}\,; \qquad \Gamma^i_{0j} = \frac{a'}{a}\,\delta^i{}_j\,; \qquad \Gamma^0_{ij} = \frac{a'}{a}\,\delta_{ij}\,; \tag{A.7}$$

$$\Gamma^i_{00} = \Gamma^0_{0i} = \Gamma^i_{jk} = 0\,. \tag{A.8}$$

The first-order perturbed connection coefficients corresponding to first-order metric perturbations in Eq. (4) are

$$\delta^{(1)}\Gamma^0_{00} = {\phi^{(1)}}'\,, \tag{A.9}$$

$$\delta^{(1)}\Gamma^0_{0i} = \partial_i\,\phi^{(1)} + \frac{a'}{a}\partial_i\,\omega^{(1)}\,, \tag{A.10}$$

$$\delta^{(1)}\Gamma^i_{00} = \frac{a'}{a}\,\partial^i\omega^{(1)} + \partial^i{\omega^{(1)}}' + \partial^i\phi^{(1)}\,, \tag{A.11}$$

$$\delta^{(1)}\Gamma^0_{ij} = -\,2\,\frac{a'}{a}\,\phi^{(1)}\,\delta_{ij} - \partial_i\partial_j\omega^{(1)} - 2\,\frac{a'}{a}\,\psi^{(1)}\,\delta_{ij} - {\psi^{(1)}}'\,\delta_{ij} - \frac{a'}{a}\,D_{ij}\chi^{(1)} + \frac{1}{2}\,D_{ij}{\chi^{(1)}}'\,, \tag{A.12}$$

$$\delta^{(1)}\Gamma^i_{0j} = -\,{\psi^{(1)}}'\delta_{ij} + \frac{1}{2}\,D_{ij}{\chi^{(1)}}'\,, \tag{A.13}$$

$$\delta^{(1)}\Gamma^i_{jk} = \partial_j\psi^{(1)}\,\gamma^i_k - \partial_k\psi^{(1)}\,\gamma^i_j + \partial^i\psi^{(1)}\,\gamma_{jk} - \frac{a'}{a}\,\partial^i\omega^{(1)}\,\gamma_{jk} + \frac{1}{2}\,\partial_j D^i_k\chi^{(1)} + \frac{1}{2}\,\partial_k D^i_j\chi^{(1)} - \frac{1}{2}\,\partial^i D_{jk}\chi^{(1)}\,. \tag{A.14}$$

At second order we get:

$$\delta^{(2)}\Gamma^0_{00} = \frac{1}{2}\,{\phi^{(2)}}' - 2\,\phi^{(1)}\,{\phi^{(1)}}' + \partial^k\phi^{(1)}\,\partial_k\omega^{(1)} + \frac{a'}{a}\partial^k\omega^{(1)}\,\partial_k\omega^{(1)} + \partial^k\omega^{(1)}\,\partial_k{\omega^{(1)}}'\,, \tag{A.15}$$

$$\begin{aligned}\delta^{(2)}\Gamma^0_{0i} = {}& \frac{1}{2}\partial_i\,\phi^{(2)} + \frac{1}{2}\,\frac{a'}{a}(\partial_i\omega^{(2)} + \omega_i^{(2)}) - 2\,\phi^{(1)}\,\partial_i\,\phi^{(1)} - 2\,\frac{a'}{a}\phi^{(1)}\,\partial_i\omega^{(1)} - {\psi^{(1)}}'\partial_i\omega^{(1)} \\ & + \frac{1}{2}\,\partial^k\omega^{(1)}\,D_{ik}{\chi^{(1)}}'\,,\end{aligned} \tag{A.16}$$

$$\begin{aligned}\delta^{(2)}\Gamma^i_{00} = {}& \frac{1}{2}\,\partial^i\,\phi^{(2)} + \frac{1}{2}\,\frac{a'}{a}(\partial^i\omega^{(2)} + \omega^{i(2)}) + \frac{1}{2}\,\left(\partial^i{\omega^{(2)}}' + \left(\omega^{i(2)}\right)'\right) + 2\,\psi^{(1)}\,\partial^i\,\phi^{(1)} - {\phi^{(1)}}'\partial^i\omega^{(1)} \\ & + 2\,\frac{a'}{a}\psi^{(1)}\,\partial^i\omega^{(1)} + 2\,\psi^{(1)}\,\partial^i{\omega^{(1)}}' - \partial_k\phi^{(1)}\,D^{ik}\chi^{(1)} - \frac{a'}{a}\partial_k\omega^{(1)}\,D^{ik}\chi^{(1)} - \partial_k{\omega^{(1)}}'D^{ik}\chi^{(1)}\,,\end{aligned} \tag{A.17}$$

$$\begin{aligned}\delta^{(2)}\Gamma^0_{ij} = {}& \Big(-\,\frac{a'}{a}\,\phi^{(2)} - \frac{1}{2}\,{\psi^{(2)}}' - \frac{a'}{a}\,\psi^{(2)} + 4\,\frac{a'}{a}\,\left(\phi^{(1)}\right)^2 + 2\,\phi^{(1)}\,{\psi^{(1)}}' + 4\,\frac{a'}{a}\,\phi^{(1)}\,\psi^{(1)} + \partial^k\omega^{(1)}\,\partial_k\psi^{(1)} \\ & - \frac{a'}{a}\partial^k\omega^{(1)}\,\partial_k\omega^{(1)}\Big)\,\delta_{ij} - \frac{1}{2}\,\partial_i\,\partial_j\omega^{(2)} + \frac{1}{4}\,\left(D_{ij}{\chi^{(2)}}' + \partial_j\,{\chi_i^{(2)}}' + \partial_i\,{\chi_j^{(2)}}' + \left(\chi_{ij}^{(2)}\right)'\right) \\ & + \frac{1}{2}\,\frac{a'}{a}\,\left(D_{ij}\chi^{(2)} + \partial_i\,\chi_j^{(2)} + \partial_j\,\chi_i^{(2)} + \chi_{ij}^{(2)}\right) - \frac{1}{4}\,\left(\partial_i\omega_j^{(2)} + \partial_j\omega_i^{(2)}\right)\end{aligned} \tag{A.18}$$

$$
\begin{aligned}
&+\; 2\,\phi^{(1)}\,\partial_i\partial_j\omega^{(1)} \;-\; \partial_i\psi^{(1)}\,\partial_j\omega^{(1)} \;-\; \partial_j\psi^{(1)}\,\partial_i\omega^{(1)} \;-\; \phi^{(1)}\,D_{ij}\chi^{(1)\prime} \;+\; \frac{1}{2}\,\partial^k\omega^{(1)}\,\partial_i D_{kj}\chi^{(1)} \\
&+\; \frac{1}{2}\,\partial^k\omega^{(1)}\,\partial_j D_{ik}\chi^{(1)} \;-\; \frac{1}{2}\,\partial^k\omega^{(1)}\,\partial_k D_{ij}\chi^{(1)}\,,
\end{aligned}
$$

$$
\begin{aligned}
\delta^{(2)}\Gamma^i_{0j} \;=&\; -\frac{1}{2}\,\psi^{(2)\prime}\delta^i{}_j \;+\; \frac{1}{4}\,\left(D^i{}_j\chi^{(2)\prime} + \partial_j\left(\chi^{i(2)}\right)^\prime + \partial^i\left(\chi^{(2)}_j\right)^\prime + \left(\chi^{i(2)}{}_j\right)^\prime\right) \;+\; \frac{1}{4}\,\left(\partial_j\omega^{i(2)} - \partial^i\omega^{(2)}_j\right) \\
&- 2\,\psi^{(1)}\,\psi^{(1)\prime}\delta^i{}_j \;-\; \partial^i\omega^{(1)}\,\partial_j\phi^{(1)} \;-\; \frac{a^\prime}{a}\partial^i\omega^{(1)}\,\partial_j\omega^{(1)} \;+\; \psi^{(1)}\,D^i{}_j\chi^{(1)\prime} \;+\; \psi^{(1)\prime}D^i{}_j\chi^{(1)} \\
&- \frac{1}{2}\,D^{ik}\chi^{(1)}\,D_{kj}\chi^{(1)\prime}\,,
\end{aligned}
\tag{A.19}
$$

$$
\begin{aligned}
\delta^{(2)}\Gamma^i_{jk} \;=&\; \frac{1}{2}\,\left(-\partial_j\,\psi^{(2)}\,\delta^i{}_k - \partial_k\,\psi^{(2)}\,\delta^i{}_j + \partial^i\,\psi^{(2)}\,\delta_{jk}\right) \;+\; \frac{1}{4}\left(\partial_j D^i{}_k\chi^{(2)} + \partial_k D^i{}_j\chi^{(2)} - \partial^i D_{jk}\chi^{(2)}\right) \\
&+ \frac{1}{2}\,\partial_j\partial_k\,\chi^{i(2)} + \frac{1}{4}\,\left(\partial_j\,\chi^{i(2)}_k + \partial_k\,\chi^{i(2)}_j - \partial^i\,\chi^{(2)}_{jk}\right) \;-\; \frac{1}{2}\,\frac{a^\prime}{a}\,\left(\partial^i\omega^{(2)} + \omega^{i(2)}\right)\,\delta_{jk} \\
&+ 2\,\psi^{(1)}\,\left(-\partial_j\,\psi^{(1)}\,\delta^i{}_k - \partial_k\,\psi^{(1)}\,\delta^i{}_j + \partial^i\,\psi^{(1)}\,\delta_{jk}\right) \;+\; 2\,\frac{a^\prime}{a}\phi^{(1)}\,\partial^i\omega^{(1)}\,\delta_{jk} \;+\; \partial^i\omega^{(1)}\,\partial_j\partial_k\omega^{(1)} \\
&+ \psi^{(1)\prime}\partial^i\omega^{(1)}\,\delta_{jk} \;+\; \psi^{(1)}\,\left(\partial_j D^i{}_k\chi^{(1)} + \partial_k D^i{}_j\chi^{(1)} - \partial^i D_{jk}\chi^{(1)}\right) \;+\; \partial_j\psi^{(1)}\,D^i{}_k\chi^{(1)} \\
&+ \partial_k\psi^{(1)}\,D^i{}_j\chi^{(1)} \;-\; \partial_m\psi^{(1)}\,D^{im}\chi^{(1)}\,\delta_{jk} \;-\; \frac{a^\prime}{a}\partial^i\omega^{(1)}\,D_{jk}\chi^{(1)} \;+\; \frac{a^\prime}{a}\partial^m\omega^{(1)}\,D^i{}_m\chi^{(1)}\,\delta_{jk} \\
&- \frac{1}{2}\,\partial^i\omega^{(1)}\,D_{jk}\chi^{(1)\prime} \;-\; \frac{1}{2}\,D^{im}\chi^{(1)}\,\partial_j D_{mk}\chi^{(1)} \;-\; \frac{1}{2}\,D^{im}\chi^{(1)}\,\partial_k D_{mj}\chi^{(1)} + \frac{1}{2}\,D^{im}\chi^{(1)}\,\partial_m D_{jk}\chi^{(1)}\,.
\end{aligned}
\tag{A.20}
$$

3. The Ricci tensor components

In a spatially flat Robertson-Walker background the components of the Ricci tensor $R_{\mu\nu}$ are given by

$$
R_{00} \;=\; -\,3\,\frac{a^{\prime\prime}}{a} \;+\; 3\left(\frac{a^\prime}{a}\right)^2; \qquad R_{0i} \;=\; 0\,; \tag{A.21}
$$

$$
R_{ij} \;=\; \left[\frac{a^{\prime\prime}}{a} \;+\; \left(\frac{a^\prime}{a}\right)^2\right]\,\delta_{ij}\,. \tag{A.22}
$$

The first-order perturbed Ricci tensor components are

$$
\delta^{(1)}R_{00} = \frac{a^\prime}{a}\partial_i\partial^i\omega^{(1)} + \partial_i\partial^i\omega^{(1)\prime} + \partial_i\partial^i\phi^{(1)} + 3\psi^{(1)\prime\prime} + 3\frac{a^\prime}{a}\psi^{(1)\prime} + 3\frac{a^\prime}{a}\phi^{(1)\prime}\,, \tag{A.23}
$$

$$
\delta^{(1)}R_{0i} = \frac{a^{\prime\prime}}{a}\partial_i\omega^{(1)} + \left(\frac{a^\prime}{a}\right)^2\partial_i\omega^{(1)} + 2\partial_i\psi^{(1)\prime} + 2\frac{a^\prime}{a}\partial_i\phi^{(1)} + \frac{1}{2}\partial_k D^k{}_i\chi^{(1)\prime}\,, \tag{A.24}
$$

$$
\begin{aligned}
\delta^{(1)}R_{ij} =& \left[-\frac{a^\prime}{a}\phi^{(1)\prime} - 5\frac{a^\prime}{a}\psi^{(1)\prime} - 2\frac{a^{\prime\prime}}{a}\phi^{(1)} - 2\left(\frac{a^\prime}{a}\right)^2\phi^{(1)} - 2\frac{a^{\prime\prime}}{a}\psi^{(1)} - 2\left(\frac{a^\prime}{a}\right)^2\psi^{(1)} - \psi^{(1)\prime\prime} + \partial_k\partial^k\psi^{(1)}\right. \\
&\left. -\; \frac{a^\prime}{a}\partial_k\partial^k\omega^{(1)}\right]\delta_{ij} - \partial_i\partial_j\omega^{(1)\prime} + \frac{a^\prime}{a}D_{ij}\chi^{(1)\prime} + \frac{a^{\prime\prime}}{a}D_{ij}\chi^{(1)} + \left(\frac{a^\prime}{a}\right)^2 D_{ij}\chi^{(1)} + \frac{1}{2}D_{ij}\chi^{(1)\prime\prime} + \partial_i\partial_j\psi^{(1)} \\
&- \partial_i\partial_j\phi^{(1)} - 2\frac{a^\prime}{a}\partial_i\partial_j\omega^{(1)} + \frac{1}{2}\partial_k\partial_i D^k{}_j\chi^{(1)} + \frac{1}{2}\partial_k\partial_j D^k{}_i\chi^{(1)} - \frac{1}{2}\partial_k\partial^k D_{ij}\chi^{(1)}\,.
\end{aligned}
\tag{A.25}
$$

At second order we obtain

$$
\begin{aligned}
\delta^{(2)}R_{00} =& +\frac{3}{2}\frac{a^\prime}{a}\;\phi^{(2)} \;+\; \frac{1}{2}\,\partial_i\,\partial^i\;\phi^{(2)} \;+\; \frac{3}{2}\frac{a^\prime}{a}\;\psi^{(2)\prime} \;+\; \frac{3}{2}\,\psi^{(2)\prime\prime} \;+\; \frac{1}{2}\,\frac{a^\prime}{a}\;\partial_k\,\partial^k\omega^{(2)} + \frac{1}{2}\,\partial_k\,\partial^k\omega^{(2)\prime} \\
&-\; 6\,\frac{a^\prime}{a}\;\phi^{(1)}\,\phi^{(1)\prime} \;-\; \partial^k\phi^{(1)}\,\partial_k\phi^{(1)} \;-\; 3\phi^{(1)\prime}\psi^{(1)\prime} \;+\; 2\,\psi^{(1)}\,\partial_i\,\partial^i\;\phi^{(1)} - \partial_k\psi^{(1)}\,\partial^k\phi^{(1)} \\
&+ 6\,\frac{a^\prime}{a}\;\psi^{(1)}\,\psi^{(1)\prime} \;+\; 6\psi^{(1)}\,\psi^{(1)\prime\prime} \;+\; 3\left(\psi^{(1)\prime}\right)^2 \;-\; \phi^{(1)\prime}\partial_i\,\partial^i\;\omega^{(1)} + \frac{a^\prime}{a}\;\partial^k\omega^{(1)}\,\partial_k\phi^{(1)}
\end{aligned}
\tag{A.26}
$$

$$+\ \frac{a''}{a}\ \partial^k\omega^{(1)}\partial_k\omega^{(1)} + \left(\frac{a'}{a}\right)^2\partial^k\omega^{(1)}\,\partial_k\omega^{(1)}\ -\ \frac{a'}{a}\partial_k\psi^{(1)}\,\partial^k\omega^{(1)} +\ 2\,\frac{a'}{a}\psi^{(1)}\,\partial_i\,\partial^i\omega^{(1)}$$
$$-\ \partial_k\psi^{(1)}\,\partial^k\omega^{(1)\prime} +\ 2\,\psi^{(1)}\,\partial_i\,\partial^i\omega^{(1)\prime}\ +\ 3\,\frac{a'}{a}\ \partial^k\omega^{(1)}\,\partial_k\omega^{(1)\prime} - \partial_k\phi^{(1)}\,\partial_i D^{ik}\chi^{(1)}$$
$$-\ \partial_i\partial_k\phi^{(1)}\,D^{ik}\chi^{(1)}\ -\frac{a'}{a}\ \partial_i\partial_k\omega^{(1)}\,D^{ik}\chi^{(1)}\ -\ \frac{a'}{a}\partial_k\omega^{(1)}\,\partial_i D^{ik}\chi^{(1)} -\ \partial_k\omega^{(1)\prime}\,\partial_i D^{ik}\chi^{(1)}$$
$$-\,\partial_i\partial_k\omega^{(1)\prime}\,D^{ik}\chi^{(1)}\ +\ \frac{1}{2}D^{ik}\chi^{(1)}\,D_{ki}\chi^{(1)\prime\prime} + \frac{1}{4}D^{ik}\chi^{(1)\prime}\,D_{ki}\chi^{(1)\prime}\ +\ \frac{1}{2}\frac{a'}{a}\ D^{ik}\chi^{(1)}\,D_{ki}\chi^{(1)\prime}\,.$$

$$\delta^{(2)}R_{0i} = +\,\frac{a'}{a}\partial_i\phi^{(2)}\ +\ \partial_i\psi^{(2)\prime}\ +\ \frac{1}{4}\partial_k\,D^k{}_i\chi^{(2)\prime}\ +\ \frac{1}{4}\partial_i\,\partial^i\,\chi_i^{(2)\prime}\ -\ \frac{1}{4}\partial_i\,\partial^i\,\omega_i^{(2)} \tag{A.27}$$
$$+\,\frac{1}{2}\frac{a''}{a}(\,\partial_i\omega^{(2)}+\omega_i^{(2)}\,) + \frac{1}{2}\Big(\frac{a'}{a}\Big)^2(\,\partial_i\omega^{(2)}+\omega_i^{(2)}\,)\ -\ 4\,\frac{a'}{a}\,\phi^{(1)}\,\partial_i\phi^{(1)} - 2\psi^{(1)\prime}\partial_i\phi^{(1)}$$
$$+\ 4\,\psi^{(1)\prime}\,\partial_i\psi^{(1)}\ +\ 4\,\psi^{(1)}\,\partial_i\psi^{(1)\prime}\ -\ 2\,\frac{a''}{a}\phi^{(1)}\partial_i\omega^{(1)}\ -\ 2\left(\frac{a'}{a}\right)^2\phi^{(1)}\,\partial_i\omega^{(1)} - \frac{a'}{a}\phi^{(1)\prime}\partial_i\omega^{(1)}$$
$$-\ \partial_i\,\partial^i\omega^{(1)}\,\partial_i\phi^{(1)}\ -\ \partial^k\omega^{(1)}\,\partial_i\partial_k\phi^{(1)}\ +\ \partial_k\phi^{(1)}\,\partial_i\partial_k\omega^{(1)}\ -\ \partial^k\omega^{(1)}\,\partial_i\partial_k\omega^{(1)\prime} - \frac{a'}{a}\partial_i\,\partial^i\omega^{(1)}\partial_i\omega^{(1)}$$
$$-\ \psi^{(1)\prime\prime}\partial_i\omega^{(1)}\ -\ 5\,\frac{a'}{a}\psi^{(1)\prime}\partial_i\omega^{(1)}\ -\ \frac{1}{2}\partial^k\phi^{(1)}\,D^{ik}\chi^{(1)\prime}\ +\ \psi^{(1)}\,\partial_k\,D^k{}_i\chi^{(1)\prime} + \psi^{(1)\prime}\,\partial_k\,D^k{}_i\chi^{(1)}$$
$$-\ \frac{1}{2}\partial_k\psi^{(1)}\,D_i^k\chi^{(1)\prime}\ +\ \partial_k\psi^{(1)\prime}\,D^k{}_i\chi^{(1)}\ +\ \frac{a'}{a}\partial^k\omega^{(1)}\,D_{ik}\chi^{(1)\prime} + \frac{1}{2}\partial^k\omega^{(1)}\,D_{ik}\chi^{(1)\prime\prime}$$
$$-\ \frac{1}{2}\partial_k D^{km}\chi^{(1)}\,D_{mi}\chi^{(1)\prime}\ -\ \frac{1}{2}D^{km}\chi^{(1)}\,\partial_k D_{mi}\chi^{(1)\prime} + \frac{1}{2}D^{km}\chi^{(1)\prime}\,\partial_i D_{mk}\chi^{(1)} + \frac{1}{4}D^{km}\chi^{(1)}\,\partial_i D_{mk}\chi^{(1)\prime}\,.$$

The purely spatial part of $\delta^{(2)}\mathcal{R}_{\mu\nu}$ is very long, and for simplicity has been divided into two parts, a diagonal part $\delta^{(2)}R^d_{ij}$ which is proportional to δ_{ij}, and a non-diagonal part R^{nd}_{ij}.

$$\delta^{(2)}R^d_{ij} = \Bigg[-\ \left(\frac{a'}{a}\right)^2\phi^{(2)}\ -\ \frac{1}{2}\frac{a'}{a}\phi^{(2)\prime}\ -\ \frac{a''}{a}\phi^{(2)}\ -\ -\frac{5}{2}\,\frac{a'}{a}\psi^{(2)\prime}\ -\ \left(\frac{a'}{a}\right)^2\psi^{(2)}\ -\ \frac{1}{2}\psi^{(2)\prime\prime} \tag{A.28}$$
$$-\ \frac{a''}{a}\psi^{(2)}\ +\ \frac{1}{2}\,\partial_i\,\partial^i\psi^{(2)}\ -\ \frac{1}{2}\frac{a'}{a}\partial_i\,\partial^i\omega^{(2)}\ +\ 4\left(\left(\frac{a'}{a}\right)^2+\frac{a''}{a}\right)\left(\phi^{(1)}\right)^2\ +\ 4\,\frac{a'}{a}\phi^{(1)}\,\phi^{(1)\prime}$$
$$+\,10\,\frac{a'}{a}\phi^{(1)}\,\psi^{(1)\prime}\ +\ 2\,\frac{a'}{a}\phi^{(1)\prime}\psi^{(1)}\ +\ \phi^{(1)\prime}\psi^{(1)\prime}\ +\ 2\,\phi^{(1)}\,\psi^{(1)\prime\prime}\ +\ 4\left(\left(\frac{a'}{a}\right)^2+\frac{a''}{a}\right)\phi^{(1)}\,\psi^{(1)}$$
$$+\,\partial_k\psi^{(1)}\,\partial^k\phi^{(1)}\ +\ \left(\psi^{(1)\prime}\right)^2\ +\ \partial_k\psi^{(1)}\,\partial^k\psi^{(1)}\ +\ 2\,\psi^{(1)}\partial_i\,\partial^i\psi^{(1)}\ +\ \frac{a'}{a}\partial_k\phi^{(1)}\,\partial^k\omega^{(1)}$$
$$+\,2\,\frac{a'}{a}\phi^{(1)}\,\partial_i\,\partial^i\omega^{(1)}\ -\left(\frac{a'}{a}\right)^2\partial_k\omega^{(1)}\,\partial^k\omega^{(1)}\ -\ \frac{a''}{a}\partial_k\omega^{(1)}\,\partial^k\omega^{(1)}\ -\ \frac{a'}{a}\partial_k\omega^{(1)}\,\partial^k\omega^{(1)\prime}$$
$$+\,3\,\frac{a'}{a}\partial_k\omega^{(1)}\,\partial^k\psi^{(1)}\ +\ 2\,\partial_k\psi^{(1)\prime}\,\partial^k\omega^{(1)}\ +\ \psi^{(1)\prime}\partial_i\,\partial^i\omega^{(1)}\ +\ \partial_k\psi^{(1)}\,\partial^k\omega^{(1)\prime} - \partial_m\psi^{(1)}\,\partial_k D^{km}\chi^{(1)}$$
$$-\,\partial_k\partial_m\psi^{(1)}\,D^{km}\chi^{(1)}\ +\ \frac{a'}{a}\partial_m\partial^k\omega^{(1)}\,D_k^m\chi^{(1)} + \frac{a'}{a}\partial^k\omega^{(1)}\,\partial_m D_k^m\chi^{(1)} - \frac{1}{2}\frac{a'}{a}D^{mk}\chi^{(1)}\,D_{km}\chi^{(1)\prime}\Bigg]\,\delta_{ij}\,,$$

$$\delta^{(2)}R^{nd}_{ij} = -\frac{1}{2}\,\partial_i\partial_j\phi^{(2)}\ +\ \frac{1}{2}\,\partial_i\partial_j\psi^{(2)}\ -\ \frac{a'}{a}\partial_i\partial_j\omega^{(2)}\ -\ \frac{1}{2}\,\partial_i\partial_j\omega^{(2)\prime}\ -\ \frac{1}{2}\frac{a'}{a}\left(\partial_i\,\omega_j^{(2)}+\partial_j\,\omega_i^{(2)}\right) \tag{A.29}$$
$$-\ \frac{1}{4}\left(\partial_i\,\omega_j^{(2)\prime}+\partial_j\,\omega_i^{(2)\prime}\right) + \frac{1}{2}\left(\left(\frac{a'}{a}\right)^2+\frac{a''}{a}\right)\left(D_{ij}\chi^{(2)}+\partial_i\chi_j^{(2)}+\partial_j\chi_i^{(2)}+\chi_{ij}^{(2)}\right)$$
$$+\ \frac{1}{2}\frac{a'}{a}\left(D_{ij}\chi^{(2)\prime}+\partial_i\chi_j^{(2)\prime}+\partial_j\chi_i^{(2)\prime}+\left(\chi_{ij}^{(2)}\right)'\right)\ +\ \frac{1}{2}\partial_k\partial_i\,D^k{}_j\chi^{(2)}\ -\ \frac{1}{4}\partial_i\,\partial^i D_{ij}\chi^{(2)}$$
$$-\ \frac{1}{4}\partial_i\,\partial^i\chi_{ij}^{(2)}\ +\ \frac{1}{4}\left(D_{ij}\chi^{(2)\prime\prime}+\partial_i\chi_j^{(2)\prime\prime}+\partial_j\chi_i^{(2)\prime\prime}+\left(\chi_{ij}^{(2)}\right)''\right)\ +\ \partial_i\phi^{(1)}\,\partial_j\phi^{(1)}$$
$$+\,2\,\phi^{(1)}\,\partial_i\partial_j\phi^{(1)}\ -\ \partial_j\phi^{(1)}\,\partial_i\psi^{(1)}\ -\ \partial_i\phi^{(1)}\,\partial_j\psi^{(1)}\ +\ 3\,\partial_i\psi^{(1)}\,\partial_j\psi^{(1)}\ +\ 2\,\psi^{(1)}\,\partial_i\partial_j\psi^{(1)}$$
$$+\,4\frac{a'}{a}\phi^{(1)}\,\partial_i\partial_j\omega^{(1)}\ +\ \phi^{(1)\prime}\,\partial_i\partial_j\omega^{(1)}\ +\ 2\,\phi^{(1)}\,\partial_i\partial_j\omega^{(1)\prime}\ +\ \partial_i\,\partial^i\omega^{(1)}\,\partial_i\partial_j\omega^{(1)}$$

$$
\begin{aligned}
&- \partial_j\partial^k\omega^{(1)}\,\partial_i\partial_k\omega^{(1)} \,-\, 2\,\frac{a'}{a}\partial_i\psi^{(1)}\,\partial_j\omega^{(1)} \,-\, 2\,\frac{a'}{a}\partial_i\omega^{(1)}\,\partial_j\psi^{(1)} \,-\, \partial_i\psi^{(1)\prime}\,\partial_j\omega^{(1)} \\
&- \partial_j\psi^{(1)\prime}\,\partial_i\omega^{(1)} \,-\, \partial_i\psi^{(1)}\,\partial_j\omega^{(1)\prime} \,-\, \partial_j\psi^{(1)}\,\partial_i\omega^{(1)\prime} \,+\, \psi^{(1)\prime}\,\partial_i\partial_j\omega^{(1)} \,-\, 2\left(\frac{a'}{a}\right)^2\phi^{(1)}\,D_{ij}\chi^{(1)} \\
&- 2\,\frac{a''}{a}\phi^{(1)}\,D_{ij}\chi^{(1)} \,-\, 2\,\frac{a'}{a}\phi^{(1)}\,D_{ij}\chi^{(1)\prime} \,-\, \frac{a'}{a}\phi^{(1)\prime}\,D_{ij}\chi^{(1)} \,-\, \frac{1}{2}\phi^{(1)\prime}\,D_{ij}\chi^{(1)\prime} \,-\, \phi^{(1)}\,D_{ij}\chi^{(1)\prime\prime} \\
&+ \frac{1}{2}\partial_k\phi^{(1)}\,\partial_i D^k{}_j\chi^{(1)} \,+\, \frac{1}{2}\partial_k\phi^{(1)}\,\partial_j D^k{}_i\chi^{(1)} \,-\, \frac{1}{2}\partial_k\phi^{(1)}\,\partial^k D_{ij}\chi^{(1)} \,-\, 3\,\frac{a'}{a}\psi^{(1)\prime}\,D_{ij}\chi^{(1)} \\
&+ \frac{1}{2}\psi^{(1)\prime}\,D_{ij}\chi^{(1)\prime} \,+\, \frac{1}{2}\partial_k\psi^{(1)}\,\partial_i D^k{}_j\chi^{(1)} \,+\, \frac{1}{2}\partial_k\psi^{(1)}\,\partial_j D^k{}_i\chi^{(1)} \,-\, \frac{3}{2}\,\partial_k\psi^{(1)}\,\partial^k D_{ij}\chi^{(1)} \\
&+ \psi^{(1)}\,\partial_k\partial_i D^k{}_j\chi^{(1)} \,+\, \psi^{(1)}\,\partial_k\partial_j D^k{}_i\chi^{(1)} \,-\, \psi^{(1)}\,\partial_k\partial^k D_{ij}\chi^{(1)} + \partial_i\psi^{(1)}\,\partial_k D^k{}_j\chi^{(1)} \\
&+ \partial_j\psi^{(1)}\,\partial_k D^k{}_i\chi^{(1)} \,+\, \partial_k\partial_i\psi^{(1)}\,D^k{}_j\chi^{(1)} \,+\, \partial_k\partial_j\psi^{(1)}\,D^k{}_i\chi^{(1)} \,+\, \frac{1}{2}\,\partial_k\partial_i\omega^{(1)}\,D^k{}_j\chi^{(1)\prime} \\
&+ \frac{1}{2}\,\partial_k\partial_j\omega^{(1)}\,D^k{}_i\chi^{(1)\prime} - \frac{1}{2}\,\partial_k\partial^k\omega^{(1)}\,D_{ij}\chi^{(1)\prime} \,+\, \frac{1}{2}\,\partial^k\omega^{(1)}\,\partial_i D_{kj}\chi^{(1)\prime} + \frac{1}{2}\,\partial^k\omega^{(1)}\,\partial_j D_{ki}\chi^{(1)\prime} \\
&- \partial^k\omega^{(1)}\,\partial_k D_{ij}\chi^{(1)\prime} \,+\, \frac{1}{2}\,\partial^k\omega^{(1)\prime}\,\partial_i D_{kj}\chi^{(1)} \,+\, \frac{1}{2}\,\partial^k\omega^{(1)\prime}\,\partial_j D_{ki}\chi^{(1)} \,-\, \frac{1}{2}\,\partial^k\omega^{(1)\prime}\,\partial_k D_{ij}\chi^{(1)} \\
&+ \frac{a'}{a}\partial^k\omega^{(1)}\,\partial_i D_{kj}\chi^{(1)} \,+\, \frac{a'}{a}\partial^k\omega^{(1)}\,\partial_j D_{ki}\chi^{(1)} \,-\, \frac{a'}{a}\partial^k\omega^{(1)}\,\partial_k D_{ij}\chi^{(1)} \,-\, \frac{a'}{a}\partial_k\partial^k\omega^{(1)}\,D_{ij}\chi^{(1)} \\
&- \frac{1}{2}\,D^k_i\chi^{(1)\prime}\,D_{kj}\chi^{(1)\prime} \,-\, \frac{1}{2}\partial_i D_{mj}\chi^{(1)}\,\partial_k D^{km}\chi^{(1)} \,-\, \frac{1}{2}\partial_j D_{mi}\chi^{(1)}\,\partial_k D^{km}\chi^{(1)} \\
&+ \frac{1}{2}\partial_m D_{ij}\chi^{(1)}\,\partial_k D^{km}\chi^{(1)} - \frac{1}{2}\partial_k\partial_i D_{mj}\chi^{(1)}\,D^{km}\chi^{(1)} \,-\, \frac{1}{2}\partial_k\partial_j D_{mi}\chi^{(1)}\,D^{km}\chi^{(1)} \\
&+ \frac{1}{2}\partial_k\partial_m D_{ij}\chi^{(1)}\,D^{km}\chi^{(1)} \,+\, \frac{1}{2}D^{km}\chi^{(1)}\,\partial_i\partial_j D_{km}\chi^{(1)} \,+\, \frac{1}{4}\partial_i D^{mk}\chi^{(1)}\,\partial_j D_{mk}\chi^{(1)}\,.
\end{aligned}
$$

4. Ricci scalar

At zeroth order the Ricci scalar R is given by

$$
R = \frac{6}{a^2}\frac{a''}{a}\,. \tag{A.30}
$$

The first-order perturbation of R is

$$
\begin{aligned}
\delta^{(1)}R = \frac{1}{a^2}\bigg(&-6\frac{a'}{a}\partial_i\partial^i\omega^{(1)} - 2\partial_i\partial^i\omega^{(1)\prime} - 2\partial_i\partial^i\phi^{(1)} - 6\psi^{(1)\prime\prime} - 6\frac{a'}{a}\phi^{(1)\prime} - 18\frac{a'}{a}\psi^{(1)\prime} \\
&- 12\frac{a''}{a}\phi^{(1)} + 4\partial_i\partial^i\psi^{(1)} + \partial_k\partial^i D^k{}_i\chi^{(1)}\bigg)\,.
\end{aligned} \tag{A.31}
$$

At second order we find

$$
\begin{aligned}
\delta^{(2)}R = &-\partial_i\,\partial^i\phi^{(2)} \,-\, 3\,\frac{a'}{a}\phi^{(2)\prime} \,-\, 6\,\frac{a''}{a}\phi^{(2)} \,+\, 2\,\partial_i\,\partial^i\psi^{(2)} \,-\, 9\,\frac{a'}{a}\psi^{(2)\prime} \,-\, 3\,\psi^{(2)\prime\prime} \,-\, \partial_i\,\partial^i\omega^{(2)\prime} \\
&- 3\,\frac{a'}{a}\partial_i\,\partial^i\omega^{(2)} \,+\, \frac{1}{2}\,\partial_k\partial_i\,D^{ki}\chi^{(2)} \,+\, 24\,\frac{a''}{a}\left(\phi^{(1)}\right)^2 \,+\, 2\,\partial_k\phi^{(1)}\,\partial^k\phi^{(1)} \,+\, 4\,\phi^{(1)}\,\partial_i\,\partial^i\phi^{(1)} \\
&+ 24\,\frac{a'}{a}\phi^{(1)}\,\phi^{(1)\prime} \,+\, 6\,\phi^{(1)\prime}\psi^{(1)\prime} \,+\, 36\,\frac{a'}{a}\phi^{(1)}\,\psi^{(1)\prime} \,+\, 2\,\partial_k\psi^{(1)}\,\partial^k\phi^{(1)} \,-\, 4\,\psi^{(1)}\,\partial_i\,\partial^i\phi^{(1)} \\
&+ 12\,\phi^{(1)}\,\psi^{(1)\prime\prime} \,-\, 12\,\psi^{(1)}\,\psi^{(1)\prime\prime} \,-\, 36\,\frac{a'}{a}\psi^{(1)\prime}\psi^{(1)} \,+\, 6\,\partial_k\psi^{(1)}\,\partial^k\psi^{(1)} \,+\, 16\,\psi^{(1)}\,\partial_i\,\partial^i\psi^{(1)} \\
&+ 6\,\frac{a'}{a}\partial^k\omega^{(1)}\,\partial_k\phi^{(1)} \,+\, 12\,\frac{a'}{a}\phi^{(1)}\,\partial_i\,\partial^i\omega^{(1)} \,+\, 4\,\phi^{(1)}\,\partial_i\,\partial^i\omega^{(1)\prime} \,+\, 2\,\phi^{(1)\prime}\partial_i\,\partial^i\omega^{(1)} \\
&- 5\,\frac{a''}{a}\partial_k\omega^{(1)}\,\partial^k\omega^{(1)} \,-\, 6\,\frac{a'}{a}\partial_k\omega^{(1)}\,\partial^k\omega^{(1)\prime} \,+\, \partial_i\,\partial^i\omega^{(1)}\,\partial_i\,\partial^i\omega^{(1)} \,-\, \partial^i\partial^k\omega^{(1)}\,\partial_i\partial_k\omega^{(1)}
\end{aligned} \tag{A.32}
$$

$$
\begin{aligned}
&+ 8\,\partial_k\omega^{(1)}\,\partial^k\psi^{(1)\prime} \,+\, 2\,\partial_k\omega^{(1)\prime}\partial^k\psi^{(1)} \,-\, 4\,\psi^{(1)}\,\partial_i\,\partial^i\omega^{(1)\prime} \,-\, 12\,\frac{a'}{a}\psi^{(1)}\,\partial_i\,\partial^i\omega^{(1)} \\
&+ 4\,\psi^{(1)\prime}\partial_i\,\partial^i\omega^{(1)} \,+\, 2\,\partial_k\phi^{(1)}\,\partial_i D^{ik}\chi^{(1)} \,+\, 2\,\partial_i\partial_k\phi^{(1)}\,D^{ik}\chi^{(1)} \,+\, 4\,\psi^{(1)}\,\partial_k\partial_i\,D^{ki}\chi^{(1)} \\
&- 2\,\partial_k\partial_i\psi^{(1)}\,D^{ik}\chi^{(1)} \,+\, 3\,\partial_k\omega^{(1)}\,\partial^i D^k{}_i\chi^{(1)\prime} \,+\, 6\,\frac{a'}{a}\partial^k\omega^{(1)}\,\partial_i D^i{}_k\chi^{(1)} \,+\, 2\,\partial_i\omega^{(1)\prime}\partial_k D^{ik}\chi^{(1)} \\
&+ 2\,\partial_k\partial_i\omega^{(1)\prime}D^{ik}\chi^{(1)} \,+\, 6\,\frac{a'}{a}\partial_k\partial_i\omega^{(1)}\,D^{ki}\chi^{(1)} \,-\, D^{ik}\chi^{(1)}\,D_{ik}\chi^{(1)\prime\prime} \,-\, \frac{3}{4}\,D^{ik}\chi^{(1)\prime}D_{ki}\chi^{(1)\prime} \\
&- 3\,\frac{a'}{a}D^{ik}\chi^{(1)}\,D_{ik}\chi^{(1)\prime} \,-\, 2\,\partial_k\partial^i\,D_{mi}\chi^{(1)}\,D^{km}\chi^{(1)} \,+\, \partial_i\,\partial^i D_{im}\chi^{(1)}\,D^{mi}\chi^{(1)} \\
&-\, \partial_k D^{km}\chi^{(1)}\,\partial^i D_{mi}\chi^{(1)} \,+\, \frac{1}{4}\,\partial^i D^{km}\chi^{(1)}\,\partial_i D_{mk}\chi^{(1)}\,.
\end{aligned}
$$

5. The Einstein tensor components

The Einstein tensor in a spatially flat Robertson-Walker background is given by

$$
G^0{}_0 = -\frac{3}{a^2}\left(\frac{a'}{a}\right)^2 , \tag{A.33}
$$

$$
G^i{}_j = -\frac{1}{a^2}\left(2\frac{a''}{a} - \left(\frac{a'}{a}\right)^2\right)\;\delta^i{}_j\,, \tag{A.34}
$$

$$
G^0{}_i = G^i{}_0 = 0\,. \tag{A.35}
$$

The first-order perturbations of the Einstein tensor components are

$$
\delta^{(1)}G^0{}_0 = \frac{1}{a^2}\left[6\left(\frac{a'}{a}\right)^2\phi^{(1)} \,+\, 6\,\frac{a'}{a}\psi^{(1)\prime} \,+\, 2\,\frac{a'}{a}\partial_i\,\partial^i\omega^{(1)} \,-\, 2\,\partial_i\,\partial^i\psi^{(1)} \,-\, \frac{1}{2}\,\partial_k\partial^i\,D^k_i\chi^{(1)}\right], \tag{A.36}
$$

$$
\delta^{(1)}G^0{}_i = \frac{1}{a^2}\left(-2\,\frac{a'}{a}\partial_i\phi^{(1)} \,-\, 2\,\partial_i\psi^{(1)\prime} \,-\, \frac{1}{2}\,\partial_k D^k{}_i\chi^{(1)\prime}\right), \tag{A.37}
$$

$$
\begin{aligned}
\delta^{(1)}G^i{}_j = \frac{1}{a^2}\Bigg[&\Bigg(2\,\frac{a'}{a}\phi^{(1)\prime} \,+\, 4\,\frac{a''}{a}\phi^{(1)} \,-\, 2\left(\frac{a'}{a}\right)^2\phi^{(1)} \,+\, \partial_i\,\partial^i\phi^{(1)} \,+\, 4\,\frac{a'}{a}\psi^{(1)\prime} \,+\, 2\,\psi^{(1)\prime\prime} \\
&-\, \partial_i\,\partial^i\psi^{(1)} \,+\, 2\,\frac{a'}{a}\partial_i\,\partial^i\omega^{(1)} \,+\, \partial_i\,\partial^i\omega^{(1)\prime} \,+\, \frac{1}{2}\partial_k\partial^m D^k{}_m\chi^{(1)}\Bigg)\,\delta^i{}_j \\
&-\, \partial^i\partial_j\phi^{(1)} \,+\, \partial^i\partial_j\psi^{(1)} \,-\, 2\,\frac{a'}{a}\partial^i\partial_j\omega^{(1)} \,-\, \partial^i\partial_j\omega^{(1)\prime} \,+\, \frac{a'}{a}D^i{}_j\chi^{(1)\prime} \,+\, \frac{1}{2}\,D^i{}_j\chi^{(1)\prime\prime} \\
&+\, \frac{1}{2}\,\partial_k\partial^i\,D^k{}_j\chi^{(1)} \,+\, \frac{1}{2}\,\partial_k\partial_j\,D^{ik}\chi^{(1)} \,-\, \frac{1}{2}\,\partial_k\partial^k\,D^i{}_j\chi^{(1)}\Bigg]\,.
\end{aligned} \tag{A.38}
$$

The second-order perturbed Einstein tensor components are given by

$$
\begin{aligned}
\delta^{(2)}G^0{}_0 = \frac{1}{a^2}\Bigg(&3\left(\frac{a'}{a}\right)^2\phi^{(2)} \,+\, 3\,\frac{a'}{a}\psi^{(2)\prime} \,-\, \partial_i\,\partial^i\psi^{(2)} \,+\, \frac{a'}{a}\partial_i\,\partial^i\omega^{(2)} \,-\, \frac{1}{4}\partial_k\partial_i\,D^{ki}\chi^{(2)} \\
&-\, 12\left(\frac{a'}{a}\right)^2\left(\phi^{(1)}\right)^2 - 12\,\frac{a'}{a}\,\phi^{(1)}\,\psi^{(1)\prime} - 3\,\partial_i\psi^{(1)}\,\partial^i\psi^{(1)} \,-\, 8\,\psi^{(1)}\,\partial_i\,\partial^i\psi^{(1)} \,+\, 12\,\frac{a'}{a}\psi^{(1)}\,\psi^{(1)\prime} \\
&-\, 3\,\left(\psi^{(1)\prime}\right)^2 \,+\, 4\,\frac{a'}{a}\,\phi^{(1)}\,\partial_i\,\partial^i\omega^{(1)} \,-\, 2\,\frac{a'}{a}\partial_k\omega^{(1)}\,\partial^k\phi^{(1)} \,-\, \frac{1}{2}\frac{a''}{a}\,\partial_k\omega^{(1)}\,\partial^k\omega^{(1)} \\
&+\, \frac{1}{2}\,\partial_i\partial_k\omega^{(1)}\,\partial^i\partial^k\omega^{(1)} \,-\, \frac{1}{2}\,\partial_k\partial^k\omega^{(1)}\,\partial_k\partial^k\omega^{(1)} \,-\, 2\,\frac{a'}{a}\partial_k\psi^{(1)}\,\partial^k\omega^{(1)} \,+\, 4\,\frac{a'}{a}\psi^{(1)}\,\partial_i\,\partial^i\omega^{(1)} \\
&-\, 2\,\partial_k\omega^{(1)}\,\partial^k\psi^{(1)\prime} \,-\, 2\psi^{(1)\prime}\partial_i\,\partial^i\omega^{(1)} \,-\, \phi^{(1)}\,\partial_i\partial^k\,D^i{}_k\chi^{(1)} \,-\, 2\,\psi^{(1)}\partial_k\partial^i\,D^k{}_i\chi^{(1)} \\
&+\, \partial_k\partial_i\psi^{(1)}\,D^{ki}\chi^{(1)} \,-\, 2\,\frac{a'}{a}\partial_i\partial_k\omega^{(1)}\,D^{ik}\chi^{(1)} \,-\, 2\,\frac{a'}{a}\partial_k\omega^{(1)}\,\partial_i\,D^{ik}\chi^{(1)} \,-\, \partial_k\omega^{(1)}\,\partial^i D^k{}_i\chi^{(1)\prime}
\end{aligned} \tag{A.39}
$$

$$
\begin{aligned}
&- \frac{1}{2}\,\partial_i\,\partial^i\,D_{mk}\chi^{(1)}\,D^{km}\chi^{(1)} \,+\, \partial_m\partial^k\,D_{ik}\chi^{(1)}\,D^{im}\chi^{(1)} \,+\, \frac{1}{2}\,\partial_k D^{km}\chi^{(1)}\,\partial^i D_{mi}\chi^{(1)} \\
&- \frac{1}{8}\,\partial^i D^{km}\chi^{(1)}\,\partial_i D_{km}\chi^{(1)} \,+\, \frac{1}{8}\,D^{ik}\chi^{(1)\prime}\,D_{ki}\chi^{(1)\prime} \,+\, \frac{a'}{a}D^{ki}\chi^{(1)}\,D_{ik}\chi^{(1)\prime}\Big)\,,
\end{aligned}
$$

$$
\begin{aligned}
\delta^{(2)}G^i{}_0 = {}& \frac{1}{a^2}\Big(\frac{a'}{a}\partial^i\phi^{(2)} \,+\, \partial^i\psi^{(2)\prime} \,+\, \frac{1}{4}\partial_k\,D^{ki}\chi^{(2)\prime} \,+\, \frac{1}{4}\partial_i\,\partial^i\chi^{i(2)\prime} \,-\, \frac{1}{4}\partial_i\,\partial^i\omega^{i(2)} \,-\, \frac{a''}{a}\partial^i\omega^{(2)} \\
&- \frac{a''}{a}\omega^{i(2)} \,+\, 2\Big(\frac{a'}{a}\Big)^2\partial^i\omega^{(2)} \,+\, 2\Big(\frac{a'}{a}\Big)^2\omega^{i(2)} \,-\, 4\,\frac{a'}{a}\phi^{(1)}\,\partial^i\phi^{(1)} \,+\, 4\,\frac{a'}{a}\,\psi^{(1)}\,\partial^i\phi^{(1)} \\
&- 2\,\psi^{(1)\prime}\partial^i\phi^{(1)} \,+\, 4\,\psi^{(1)\prime}\partial^i\psi^{(1)} \,+\, 8\,\psi^{(1)}\,\partial^i\psi^{(1)\prime} \,-\, \partial^i\phi^{(1)}\,\partial_i\,\partial^i\omega^{(1)} \,-\, \partial^k\omega^{(1)}\,\partial^i\partial_k\phi^{(1)} \\
&+ \partial_i\,\partial^i\phi^{(1)}\,\partial^i\omega^{(1)} \,+\, \partial^i\partial_k\omega^{(1)}\,\partial^k\phi^{(1)} \,+\, 4\,\frac{a''}{a}\phi^{(1)}\,\partial^i\omega^{(1)} \,-\, 8\Big(\frac{a'}{a}\Big)^2\phi^{(1)}\,\partial^i\omega^{(1)} \\
&+ 2\,\frac{a'}{a}\phi^{(1)\prime}\partial^i\omega^{(1)} \,+\, \partial_i\,\partial^i\omega^{(1)\prime}\,\partial^i\omega^{(1)} \,-\, \partial^k\omega^{(1)}\,\partial^i\partial_k\omega^{(1)\prime} \,+\, 2\,\psi^{(1)\prime\prime}\partial^i\omega^{(1)} \\
&+\; 8\,\Big(\frac{a'}{a}\Big)^2\psi^{(1)}\,\partial^i\omega^{(1)} \,-\, 4\,\frac{a''}{a}\,\psi^{(1)}\,\partial^i\omega^{(1)} \,-\, 2\,\frac{a'}{a}\psi^{(1)\prime}\partial^i\omega^{(1)} \,-\, \frac{1}{2}\,\partial^k\phi^{(1)}\,D^i{}_k\chi^{(1)\prime} \\
&- 2\,\frac{a'}{a}\,\partial_k\phi^{(1)}\,D^{ki}\chi^{(1)} \,-\, \frac{1}{2}\,\partial_k\psi^{(1)}\,D^{ki}\chi^{(1)\prime} \,+\, 2\,\psi^{(1)}\,\partial_k D^{ki}\chi^{(1)\prime} \,+\, \psi^{(1)\prime}\partial_k D^{ki}\chi^{(1)} \\
&- \partial_k\psi^{(1)\prime}D^{ki}\chi^{(1)} \,+\, \frac{1}{2}\,\partial^k\omega^{(1)}\,D^i{}_k\chi^{(1)\prime\prime} \,+\, \frac{a'}{a}\partial^k\omega^{(1)}\,D^i{}_k\chi^{(1)\prime} \,-\, 4\Big(\frac{a'}{a}\Big)^2\partial_k\omega^{(1)}\,D^{ik}\chi^{(1)} \\
&+ 2\frac{a''}{a}\,\partial_k\omega^{(1)}\,D^{ik}\chi^{(1)} \,-\, \frac{1}{2}\,\partial_k D^{km}\chi^{(1)}\,D^i{}_m\chi^{(1)\prime} \,-\, \frac{1}{2}\,\partial_k D^i{}_m\chi^{(1)\prime}\,D^{km}\chi^{(1)} \\
&+ \frac{1}{4}\partial^i D_{mk}\chi^{(1)}\,D^{km}\chi^{(1)\prime} \,+\, \frac{1}{2}\,\partial^i D_{mk}\chi^{(1)\prime}\,D^{km}\chi^{(1)} \,-\, \frac{1}{2}\,D^{ik}\chi^{(1)}\,\partial_m D^m_k\chi^{(1)\prime}\Big)\,,
\end{aligned}
\tag{A.40}
$$

$$
\begin{aligned}
\delta^{(2)}G^0{}_i = {}& \frac{1}{a^2}\Big(-\,\frac{a'}{a}\partial_i\phi^{(2)} \,-\, \partial_i\psi^{(2)\prime} \,-\, \frac{1}{4}\partial_k\,D^k_{\ i}\chi^{(2)\prime} \,-\, \frac{1}{4}\partial_i\,\partial^i\chi_i^{(2)\prime} \,+\, \frac{1}{4}\partial_i\,\partial^i\omega_i^{(2)} \,+ 8\frac{a'}{a}\phi^{(1)}\partial_i\phi^{(1)} \\
&+\; 4\,\phi^{(1)}\,\partial_i\psi^{(1)\prime} + 2\,\psi^{(1)\prime}\partial_i\phi^{(1)} \,-\, 4\,\psi^{(1)\prime}\partial_i\psi^{(1)} \,-\, 4\,\psi^{(1)}\,\partial_i\psi^{(1)\prime} \,+\, \partial_i\phi^{(1)}\,\partial_i\,\partial^i\omega^{(1)} \\
&-\; \partial_i\partial_k\omega^{(1)}\,\partial^k\phi^{(1)} + 8\frac{a''}{a}\phi^{(1)}\,\partial_i\omega^{(1)} \,-\, 4\Big(\frac{a'}{a}\Big)^2\phi^{(1)}\,\partial_i\omega^{(1)} \,-\, 2\,\frac{a'}{a}\,\partial^k\omega^{(1)}\,\partial_i\partial_k\omega^{(1)} \,+\, \partial_i\,\partial^i\psi^{(1)}\,\partial_i\omega^{(1)} \\
&+ \partial^k\omega^{(1)}\;\partial_i\partial_k\psi^{(1)} \,-\, \partial_k\phi^{(1)}\,D^k_{\ i}\chi^{(1)\prime} \,+\, \frac{1}{2}\,\partial^k\phi^{(1)}\,D_{ik}\chi^{(1)\prime} \,-\, \psi^{(1)}\,\partial_k D^k_{\ i}\chi^{(1)\prime} \\
&+\; \frac{1}{2}\,\partial_k\psi^{(1)}\,D^k_{\ i}\chi^{(1)\prime} \,-\, \psi^{(1)\prime}\partial_k D^k_{\ i}\chi^{(1)} \,-\, \partial_k\psi^{(1)\prime}D^k_{\ i}\chi^{(1)} \,+\, \partial_i\omega^{(1)}\,\partial_k\partial^m\,D^k_{\ m}\chi^{(1)} \\
&- 2\,\frac{a''}{a}\,\partial^k\omega^{(1)}\,D_{ik}\chi^{(1)} \,+\, \Big(\frac{a'}{a}\Big)^2\partial^k\omega^{(1)}\,D_{ik}\chi^{(1)} \,+\, \partial^k\omega^{(1)}\,\partial_m\partial_i\,D^m_k\chi^{(1)} \\
&-\; \frac{1}{2}\,\partial^m\omega^{(1)}\,\partial_k\partial^k\,D_{im}\chi^{(1)} \,+\, \frac{1}{2}\,\partial_k D^{km}\chi^{(1)}\,D_{im}\chi^{(1)\prime} \,+\, \frac{1}{2}\,\partial_k D_{im}\chi^{(1)\prime}\,D^{km}\chi^{(1)} \\
&- \frac{1}{4}\partial_i D_{mk}\chi^{(1)}\,D^{km}\chi^{(1)\prime} \,-\, \frac{1}{2}\,\partial_i D_{mk}\chi^{(1)\prime}\,D^{km}\chi^{(1)}\Big)\,,
\end{aligned}
\tag{A.41}
$$

$$
\begin{aligned}
\delta^{(2)}G^{d}{}^i_{\ j} = {}& \frac{1}{a^2}\Big(+\,\frac{1}{2}\partial_i\,\partial^i\phi^{(2)} \,+\, \frac{a'}{a}\phi^{(2)\prime} \,+\, 2\,\frac{a''}{a}\,\phi^{(2)} \,-\, \Big(\frac{a'}{a}\Big)^2\phi^{(2)} \,-\, \frac{1}{2}\partial_i\,\partial^i\psi^{(2)} \,+\, \psi^{(2)\prime\prime} \\
&+\; 2\,\frac{a'}{a}\,\psi^{(2)\prime} \,+\, \frac{a'}{a}\,\partial_i\,\partial^i\omega^{(2)} \,+\, \frac{1}{2}\partial_i\,\partial^i\omega^{(2)\prime} \,-\, \frac{1}{4}\,\partial_k\partial_i\,D^{ki}\chi^{(2)} \,+\, 4\,\Big(\frac{a'}{a}\Big)^2\,\Big(\phi^{(1)}\Big)^2 \\
&- 8\,\frac{a''}{a}\,\Big(\phi^{(1)}\Big)^2 \,-\, 8\,\frac{a'}{a}\,\phi^{(1)}\,\phi^{(1)\prime} \,-\, \partial_k\phi^{(1)}\,\partial^k\phi^{(1)} \,-\, 2\phi^{(1)}\,\partial_i\,\partial^i\phi^{(1)} \,-\, 4\,\phi^{(1)}\,\psi^{(1)\prime\prime} \\
&- 2\,\phi^{(1)\prime}\psi^{(1)\prime} \,-\, 8\,\frac{a'}{a}\,\phi^{(1)}\,\psi^{(1)\prime} \,-\, 2\,\partial_k\psi^{(1)}\,\partial^k\psi^{(1)} \,-\, 4\,\psi^{(1)}\,\partial_i\,\partial^i\psi^{(1)} \,+\, \Big(\psi^{(1)\prime}\Big)^2 \,+\, 8\,\frac{a'}{a}\,\psi^{(1)}\,\psi^{(1)\prime} \\
&+ 4\,\psi^{(1)}\,\psi^{(1)\prime\prime} \,+\, 2\,\psi^{(1)}\,\partial_i\,\partial^i\phi^{(1)} \,-\, \phi^{(1)\prime}\partial_i\,\partial^i\omega^{(1)} \,-\, 2\,\phi^{(1)}\,\partial_i\,\partial^i\omega^{(1)\prime} \,-\, 2\,\frac{a'}{a}\,\partial_k\omega^{(1)}\,\partial^k\phi^{(1)} \\
&- 4\,\frac{a'}{a}\,\phi^{(1)}\,\partial_i\,\partial^i\omega^{(1)} \,+\, \frac{3}{2}\frac{a''}{a}\partial_k\omega^{(1)}\,\partial^k\omega^{(1)} \,-\, \Big(\frac{a'}{a}\Big)^2\partial_k\omega^{(1)}\,\partial^k\omega^{(1)} \,+\, 2\,\frac{a'}{a}\partial_k\omega^{(1)}\,\partial^k\omega^{(1)\prime} \\
&- \frac{1}{2}\,\partial_i\,\partial^i\omega^{(1)}\,\partial_i\,\partial^i\omega^{(1)} \,+\, \frac{1}{2}\,\partial^m\partial^k\omega^{(1)}\,\partial_m\partial_k\omega^{(1)} \,+\, 4\,\frac{a'}{a}\,\psi^{(1)}\,\partial_i\,\partial^i\omega^{(1)} \,+\, 2\,\psi^{(1)}\,\partial_i\,\partial^i\omega^{(1)\prime}
\end{aligned}
\tag{A.42}
$$

$$
\begin{aligned}
&- 2\,\partial_k\omega^{(1)}\,\partial^k\psi^{(1)\prime} - \psi^{(1)\prime}\partial_i\,\partial^i\omega^{(1)} - \partial_k\partial_m\phi^{(1)}\,D^{km}\chi^{(1)} - \partial_k\phi^{(1)}\,\partial_m D^{mk}\chi^{(1)} \\
&- \partial_k\psi^{(1)}\,\partial_m D^{mk}\chi^{(1)} - \frac{3}{2}\,\partial_k\omega^{(1)}\,\partial^i D^k{}_i\chi^{(1)\prime} - \partial_k\omega^{(1)\prime}\,\partial_m D^{mk}\chi^{(1)} \\
&- \partial_k\partial_m\omega^{(1)\prime}\,D^{km}\chi^{(1)} - 2\,\frac{a'}{a}\partial^k\omega^{(1)}\,\partial_m D^m_k\chi^{(1)} - 2\,\frac{a'}{a}\partial_m\partial^k\omega^{(1)}\,D^m_k\chi^{(1)} \\
&+ \frac{3}{4}\,\partial_k\partial^l\,D_{ml}\chi^{(1)}\,D^{km}\chi^{(1)} - \frac{1}{2}\,\partial_i\,\partial^i\,D_{ml}\chi^{(1)}\,D^{ml}\chi^{(1)} + \frac{1}{4}\,\partial_m\partial^k\,D_{lk}\chi^{(1)}\,D^{lm}\chi^{(1)} \\
&+ \frac{1}{2}\,\partial_k D_{km}\chi^{(1)}\,\partial^l D^{ml}\chi^{(1)} - \frac{1}{8}\,\partial^l D_{km}\chi^{(1)}\,\partial_l D^{km}\chi^{(1)} + \frac{1}{2}\,D^{mk}\chi^{(1)}\,D_{mk}\chi^{(1)\prime\prime} \\
&+ \frac{3}{8}\,D^{mk}\chi^{(1)\prime}\,D_{mk}\chi^{(1)\prime} + \frac{a'}{a}\,D^{mk}\chi^{(1)}\,D_{km}\chi^{(1)\prime}\Big)\delta^i{}_j\,.
\end{aligned}
$$

$$
\begin{aligned}
\delta^{(2)}G^{nd\,i}{}_j = {}& \frac{1}{a^2}\Big[- \frac{1}{2}\,\partial^i\partial_j\phi^{(2)} + \frac{1}{2}\,\partial^i\partial_j\psi^{(2)} - \frac{a'}{a}\partial^i\partial_j\omega^{(2)} - \frac{1}{2}\,\partial^i\partial_j\omega^{(2)\prime} - \frac{1}{2}\frac{a'}{a}\left(\partial^i\,\omega^{(2)}_j + \partial_j\,\omega^{i(2)}\right) \qquad \text{(A.43)} \\
&- \frac{1}{4}\left(\partial^i\,\omega^{(2)\prime}_j + \partial_j\,\omega^{i(2)\prime}\right) + \frac{1}{2}\frac{a'}{a}\left(D^i{}_j\chi^{(2)\prime} + \partial^i\chi^{(2)\prime}_j + \partial_j\chi^{i(2)\prime} + \chi^{i(2)\prime}_{\ j}\right) + \frac{1}{2}\partial_k\partial^i\,D^k{}_j\chi^{(2)} \\
&- \frac{1}{4}\partial_i\,\partial^i D^i{}_j\chi^{(2)} - \frac{1}{4}\partial_i\,\partial^i\chi^{i(2)}_{\ j} + \frac{1}{4}\left(D^i{}_j\chi^{(2)\prime\prime} + \partial^i\chi^{(2)\prime\prime}_j + \partial_j\chi^{i(2)\prime\prime} + \chi^{i(2)\prime\prime}_{\ j}\right) + \partial^i\phi^{(1)}\,\partial_j\phi^{(1)} \\
&+ 2\,\phi^{(1)}\,\partial^i\partial_j\phi^{(1)} - 2\,\psi^{(1)}\partial^i\partial_j\phi^{(1)} - \partial_j\phi^{(1)}\,\partial^i\psi^{(1)} - \partial^i\phi^{(1)}\,\partial_j\psi^{(1)} + 3\,\partial^i\psi^{(1)}\,\partial_j\psi^{(1)} + 4\,\psi^{(1)}\,\partial^i\partial_j\psi^{(1)} \\
&+ 2\,\frac{a'}{a}\,\partial^i\omega^{(1)}\,\partial_j\phi^{(1)} + 4\frac{a'}{a}\phi^{(1)}\,\partial^i\partial_j\omega^{(1)} + \phi^{(1)\prime}\,\partial^i\partial_j\omega^{(1)} + 2\,\phi^{(1)}\,\partial^i\partial_j\omega^{(1)\prime} + \partial_i\,\partial^i\omega^{(1)}\,\partial^i\partial_j\omega^{(1)} \\
&- \partial_j\partial^k\omega^{(1)}\,\partial^i\partial_k\omega^{(1)} - 2\,\frac{a'}{a}\partial^i\psi^{(1)}\,\partial_j\omega^{(1)} - 2\,\frac{a'}{a}\partial^i\omega^{(1)}\,\partial_j\psi^{(1)} - \partial^i\psi^{(1)\prime}\,\partial_j\omega^{(1)} + \partial_j\psi^{(1)\prime}\,\partial^i\omega^{(1)} \\
&- \partial^i\psi^{(1)}\,\partial_j\omega^{(1)\prime} - \partial_j\psi^{(1)}\,\partial^i\omega^{(1)\prime} - 2\,\psi^{(1)}\,\partial^i\partial_j\omega^{(1)\prime} + \psi^{(1)\prime}\,\partial^i\partial_j\omega^{(1)} - 4\,\frac{a'}{a}\,\psi^{(1)}\,\partial^i\partial_j\omega^{(1)} \\
&- 2\,\frac{a'}{a}\phi^{(1)}\,D^i{}_j\chi^{(1)\prime} - \frac{1}{2}\phi^{(1)\prime}\,D^i{}_j\chi^{(1)\prime} - \phi^{(1)}\,D^i{}_j\omega^{(1)\prime\prime} + \frac{1}{2}\partial_k\phi^{(1)}\,\partial^i D^k{}_j\chi^{(1)} + \frac{1}{2}\partial_k\phi^{(1)}\,\partial_j D^{ki}\chi^{(1)} \\
&- \frac{1}{2}\partial_k\phi^{(1)}\,\partial^k D^i{}_j\chi^{(1)} + \partial_j\partial_k\phi^{(1)}\,D^{ki}\chi^{(1)} + \frac{1}{2}\psi^{(1)\prime}\,D^i{}_j\chi^{(1)\prime} + \psi^{(1)\prime\prime}D^i{}_j\chi^{(1)} + 2\,\frac{a'}{a}\psi^{(1)\prime}\,D^i{}_j\chi^{(1)} \\
&+ \frac{1}{2}\partial_k\psi^{(1)}\,\partial^i D^k{}_j\chi^{(1)} + 2\,\frac{a'}{a}\,\psi^{(1)}\,D^i{}_j\chi^{(1)\prime} + \psi^{(1)}\,D^i{}_j\omega^{(1)\prime\prime} + \frac{1}{2}\partial_k\psi^{(1)}\,\partial_j D^{ki}\chi^{(1)} - \frac{3}{2}\,\partial_k\psi^{(1)}\,\partial^k D^i{}_j\chi^{(1)} \\
&+ 2\,\psi^{(1)}\,\partial_k\partial^i D^k{}_j\chi^{(1)} + 2\,\psi^{(1)}\,\partial_k\partial_j D^{ki}\chi^{(1)} - 2\,\psi^{(1)}\,\partial_k\partial^k D^i{}_j\chi^{(1)} - \partial_i\,\partial^i\psi^{(1)}\,D^i{}_j\chi^{(1)} + \partial^i\psi^{(1)}\,\partial_k D^k{}_j\chi^{(1)} \\
&+ \partial_j\psi^{(1)}\,\partial_k D^{ki}\chi^{(1)} + \partial_k\partial^i\psi^{(1)}\,D^k{}_j\chi^{(1)} + \frac{1}{2}\,\partial^i\omega^{(1)}\,\partial_k D^k{}_j\chi^{(1)\prime} + \frac{1}{2}\,\partial_k\partial^i\omega^{(1)}\,D^k{}_j\chi^{(1)\prime} \\
&+ \frac{1}{2}\,\partial_k\partial_j\omega^{(1)}\,D^{ki}\chi^{(1)\prime} - \frac{1}{2}\,\partial_k\partial^k\omega^{(1)}\,D^i_j\chi^{(1)\prime} + \frac{1}{2}\,\partial^k\omega^{(1)}\,\partial^i D_{kj}\chi^{(1)\prime} + \frac{1}{2}\,\partial^k\omega^{(1)}\,\partial_j D^i{}_k\chi^{(1)\prime} \\
&- \partial^k\omega^{(1)}\,\partial_k D^i{}_j\chi^{(1)\prime} + \frac{1}{2}\,\partial^k\omega^{(1)\prime}\,\partial^i D_{kj}\chi^{(1)} + \frac{1}{2}\,\partial^k\omega^{(1)\prime}\,\partial_j D^i{}_k\chi^{(1)} - \frac{1}{2}\,\partial^k\omega^{(1)\prime}\,\partial_k D^i{}_j\chi^{(1)} \\
&+ \partial_k\partial_j\omega^{(1)\prime}\,D^{ik}\chi^{(1)} + \frac{a'}{a}\partial^k\omega^{(1)}\,\partial^i D_{kj}\chi^{(1)} + \frac{a'}{a}\partial^k\omega^{(1)}\,\partial_j D^i{}_k\chi^{(1)} - \frac{a'}{a}\partial^k\omega^{(1)}\,\partial_k D^i{}_j\chi^{(1)} \\
&+ 2\,\frac{a'}{a}\,\partial_k\partial_j\omega^{(1)}\,D^{ik}\chi^{(1)} - \frac{1}{2}\,D^{ki}\chi^{(1)\prime}\,D_{kj}\chi^{(1)\prime} - \frac{1}{2}\partial^i D_{mj}\chi^{(1)}\,\partial_k D^{km}\chi^{(1)} \\
&- \frac{1}{2}\partial_j D^i{}_m\chi^{(1)}\,\partial_k D^{km}\chi^{(1)} + \frac{1}{2}\partial_m D^i{}_j\chi^{(1)}\,\partial_k D^{km}\chi^{(1)} - \frac{1}{2}\partial_k\partial^i D_{mj}\chi^{(1)}\,D^{km}\chi^{(1)} \\
&- \frac{1}{2}\partial_k\partial_j D^i{}_m\chi^{(1)}\,D^{km}\chi^{(1)} + \frac{1}{2}\partial_k\partial_m D^i{}_j\chi^{(1)}\,D^{km}\chi^{(1)} + \frac{1}{2}D^{km}\chi^{(1)}\,\partial^i\partial_j D_{km}\chi^{(1)} \\
&+ \frac{1}{4}\partial^i D^{mk}\chi^{(1)}\,\partial_j D_{mk}\chi^{(1)} - \partial_k\partial^m\,D^k{}_m\chi^{(1)}\,D^i{}_j\chi^{(1)} - \frac{a'}{a}\,D_{kj}\chi^{(1)\prime}D^{ik}\chi^{(1)} \\
&- \frac{1}{2}\,D_{kj}\omega^{(1)\prime\prime}D^{ki}\chi^{(1)} - \partial_m\partial_k\,D^m_j\chi^{(1)}\,D^{ki}\chi^{(1)} + \frac{1}{2}\,\partial_m\partial^m\,D_{kj}\chi^{(1)}\,D^{ki}\chi^{(1)}\Big]\,,
\end{aligned}
$$

where $\delta^{(2)}G^{d\,i}{}_j$ stands for the diagonal part of $\delta^{(2)}G^i_j$, which is proportional to $\delta^i{}_j$, and $\delta^{(2)}G^{nd\,i}{}_j$ is the non-diagonal contribution.

APPENDIX B: PERTURBING THE KLEIN-GORDON EQUATION

In the homogeneous background the Klein-Gordon equation for the scalar field φ is

$$\varphi_0'' + 2\frac{a'}{a}\varphi_0' = -\frac{\partial V}{\partial \varphi}a^2 \tag{B.1}$$

The perturbed Klein-Gordon equation at first-order is

$$\delta^{(1)}\varphi'' + 2\,\frac{a'}{a}\delta^{(1)}\varphi' - \partial_i\,\partial^i\delta^{(1)}\varphi - \phi^{(1)'}\varphi_0' - 3\,\psi^{(1)'}\varphi_0' - \partial_i\,\partial^i\omega^{(1)}\,\varphi_0' = -\delta^{(1)}\varphi\,\frac{\partial^2 V}{\partial\varphi^2}\,a^2 - 2\,\phi^{(1)}\,\frac{\partial V}{\partial\varphi}. \tag{B.2}$$

At second order we get

$$\begin{aligned}
&-\frac{1}{2}\delta^{(2)}\varphi'' - \frac{a'}{a}\delta^{(2)}\varphi' + \frac{1}{2}\partial_i\,\partial^i\delta^{(2)}\varphi + \phi^{(2)}\,\varphi_0'' + 2\,\frac{a'}{a}\phi^{(2)}\,\varphi_0' + \frac{1}{2}\,\phi^{(2)'}\varphi_0' \\
&+\frac{3}{2}\,\psi^{(2)'}\,\varphi_0' + \frac{1}{2}\,\partial_i\,\partial^i\omega^{(2)}\,\varphi_0' - 4\,\left(\phi^{(1)}\right)^2\,\varphi_0'' - 8\,\frac{a'}{a}\left(\phi^{(1)}\right)^2\,\varphi_0' - 4\,\phi^{(1)}\,\phi^{(1)'}\varphi_0' \\
&+2\,\phi^{(1)}\,\delta^{(1)}\varphi'' + \phi^{(1)'}\delta^{(1)}\varphi' + 4\,\frac{a'}{a}\phi^{(1)}\,\delta^{(1)}\varphi' + \partial^k\phi^{(1)}\,\partial_k\delta^{(1)}\varphi - 6\,\phi^{(1)}\,\psi^{(1)'}\varphi_0' \\
&+6\,\psi^{(1)}\,\psi^{(1)'}\varphi_0' + 3\,\psi^{(1)'}\delta^{(1)}\varphi' - \partial^k\psi^{(1)}\,\partial_k\delta^{(1)}\varphi + 2\,\psi^{(1)}\,\partial_i\,\partial^i\delta^{(1)}\varphi \\
&-2\,\phi^{(1)}\,\partial_i\,\partial^i\omega^{(1)}\,\varphi_0' - \partial^k\omega^{(1)}\,\partial_k\phi^{(1)}\,\varphi_0' - \partial^k\omega^{(1)}\,\partial_k\psi^{(1)}\,\varphi_0' + 2\,\psi^{(1)}\,\partial_i\,\partial^i\omega^{(1)}\,\varphi_0' \\
&+\partial^k\omega^{(1)}\,\partial_k\omega^{(1)}\,\varphi_0'' + 2\,\frac{a'}{a}\partial^k\omega^{(1)}\,\partial_k\omega^{(1)}\,\varphi_0' + \partial_k\omega^{(1)}\,\partial^k\omega^{(1)'}\varphi_0' + 2\,\partial^k\omega^{(1)}\,\partial_k\,\delta^{(1)}\varphi' \\
&+2\,\frac{a'}{a}\partial^k\omega^{(1)}\,\partial_k\,\delta^{(1)}\varphi + \partial_i\,\partial^i\omega^{(1)}\,\delta^{(1)}\varphi' + \partial^k\omega^{(1)'}\partial_k\delta^{(1)}\varphi - \partial^k\omega^{(1)}\,\partial_i D^i{}_k\chi^{(1)}\,\varphi_0' \\
&-\partial_i\partial_k\omega^{(1)}\,D^{ik}\chi^{(1)}\,\varphi_0' - \partial_i\partial_k\,\delta^{(1)}\varphi\,D^{ik}\chi^{(1)} - \partial_k\,\delta^{(1)}\varphi\,\partial^i D^k{}_i\chi^{(1)} \\
&+\frac{1}{2}\,D^{ik}\chi^{(1)}\,D_{ki}\chi^{(1)'}\,\varphi_0' = \frac{1}{2}\frac{\partial^2 V}{\partial\varphi^2}\,\delta^{(2)}\varphi\,a^2 + \frac{1}{2}\,\frac{\partial^3 V}{\partial\varphi^3}(\delta^{(1)}\varphi)^2\,a^2\,.
\end{aligned} \tag{B.3}$$

To obtain the Klein-Gordon equation in the longitudinal gauge of Eq. (28) one can simply set $\chi^{(1)} = \chi^{(2)} = 0$, and $\phi^{(1)} = \psi^{(1)}$. Thus at first-order we find

$$\delta^{(1)}\varphi'' + 2\,\frac{a'}{a}\delta^{(1)}\varphi' - \partial_i\,\partial^i\delta^{(1)}\varphi - 4\phi^{(1)'}\varphi_0' = -\delta^{(1)}\varphi\,\frac{\partial^2 V}{\partial\varphi^2}\,a^2 - 2\,\phi^{(1)}\,\frac{\partial V}{\partial\varphi}\,, \tag{B.4}$$

while at second order the equation is

$$\begin{aligned}
&+\frac{1}{2}\delta^{(2)}\varphi'' + \frac{a'}{a}\delta^{(2)}\varphi' - \frac{1}{2}\partial_i\,\partial^i\delta^{(2)}\varphi - \phi^{(2)}\,\varphi_0'' - 2\,\frac{a'}{a}\phi^{(2)}\,\varphi_0' - \frac{1}{2}\,\phi^{(2)'}\varphi_0' \\
&-\frac{3}{2}\,\psi^{(2)'}\,\varphi_0' - 4\,\phi^{(1)}\,\phi^{(1)'}\varphi_0' - 4\,\phi^{(1)'}\delta^{(1)}\varphi' - 4\,\phi^{(1)}\,\partial_i\,\partial^i\delta^{(1)}\varphi = \\
&-2\phi^{(1)}\,\delta^{(1)}\varphi\frac{\partial^2 V}{\partial\varphi^2}\,a^2 - \frac{1}{2}\delta^{(2)}\varphi\,\frac{\partial^2 V}{\partial\varphi^2}\,a^2 - \frac{1}{2}\,(\delta^{(1)}\varphi)^2\,\frac{\partial^3 V}{\partial\varphi^3}\,a^2\,,
\end{aligned} \tag{B.5}$$

where we have used the background equation (B.1) and the first-order perturbed equation (B.4) to simplify some terms.

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